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Wave breaking and global existence for the generalized periodic two-component Hunter–Saxton system

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In this paper, we study the wave-breaking phenomena and global existence for the generalized two-component Hunter–Saxton system in the periodic setting. We first establish local well-posedness for the generalized two-component Hunter–Saxton system. We obtain a wave-breaking criterion for solutions and results of wave-breaking solutions with certain initial profiles. We also determine the exact blow-up rate of strong solutions. Finally, we give a sufficient condition for global solutions.

1. Introduction

In this paper, we are concerned with the initial-value problem associated with the generalized periodic two-component Hunter–Saxton system

$$\begin{cases} u_{txx} + 2\sigma u_x u_{xx} + \sigma u u_{xxx} - \rho\rho_x + A u_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), \quad \rho(t, x+1) = \rho(t, x), & t \geqslant 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where $\sigma \in \mathbb{R}$ is the new free parameter, and $A \geqslant 0$. System (1.1) is the short-wave (or high-frequency) limit

$$(t,x) \mapsto (\epsilon t, \epsilon x), \quad \epsilon \to 0$$

of the generalized two-component Camassa–Holm system (gCH2) established in [5] which can be derived from shallow water theory with nonzero constant vorticity by using Ivanov's modeling approach [20],

$$\begin{cases} m_t - Au_x + \sigma(2mu_x + um_x) + 3(1-\sigma)uu_x + \rho\rho_x = 0, \quad m = u - u_{xx}, \\ \rho_t + (\rho u)_x = 0, \end{cases}$$

or equivalently, in terms of u and ρ,

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases}$$

where $u(t,x)$ represents the horizontal velocity of the fluid, and $\rho(t,x)$ is related to the free surface elevation from equilibrium (or scalar density) with the boundary assumption, $u \to 0$, $\rho \to 1$ as $|x| \to \infty$. The parameter $A > 0$ characterizes a linear underlying shear flow so that the two-component CH system models wave–current interactions. The real dimensionless constant σ is a parameter which provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due to stretching. The main motivation for seeking and studying such systems lies in capturing nonlinear phenomena such as wave-breaking and traveling waves [10,33,34] which are not exhibited by small-amplitude models [10]. Another heuristic motivation for studying the generalized Hunter–Saxton system comes from its analogy with hydrodynamically relevant equations (e.g., the incompressible vorticity equation in three space dimensions), in which the interplay of convection (uu_{xxx}) and stretching ($u_x u_{xx}$) is crucial for the creation of spontaneous singularities or boundedness [36]. Similarly to [35,37], the size of the stretching parameter σ will illustrate the inherent importance of the convection term in delaying or depleting finite-time blow-up.

When $(\sigma, A) = (1, 0)$, (1.1) becomes the two-component Hunter–Saxton system

$$\begin{cases} u_{txx} + 2u_x u_{xx} + uu_{xxx} - \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.2}$$

which is a generalization of the Hunter–Saxton equation modeling the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal (see Hunter and Saxton [17] for a derivation, and also [1–3,40,45]). Furthermore, the two-component Hunter–Saxton system is a particular case of the Gurevich–Zybin system [16] pertaining to nonlinear one-dimensional dynamics of dark matter as well as nonlinear ion-acoustic waves (cf. [39] and the references therein).

It was noted by Constantin and Ivanov [7] that the Hunter–Saxton system is formally integrable with a bi-Hamiltonian structure – it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ:

$$\Psi_{xx} = \left(-\zeta^2\rho^2 + \zeta m\right)\Psi,$$
$$\Psi_t = \left(\frac{1}{2\zeta} - u\right)\Psi_x + \frac{1}{2}u_x\Psi, \quad m = -u_{xx},$$

and allows for peakon solutions. Moreover, Lenells and Lechtenfeld [30] showed that it can be interpreted as the Euler equation on the superconformal algebra of contact vector fields, which is in accordance with the well-known geometric interpretation of the Hunter–Saxton equation as the geodesic flow of the right-invariant $\dot{H}^1(\mathbb{S})$ metric on the space of orientation preserving circle diffeomorphisms modulo rigid rotations [24,27,29,30] (see also [8,9,11,26,32] for related geodesic flow equations). Its local well-posedness, global existence and blow-up phenomena were discussed recently in [42]. Moreover, Wu and Wunsch [41], and Liu and Yin [31] gave sufficient conditions for the global existence of strong solutions to the Hunter–Saxton system. On the other hand, Escher [13] gives geometric meaning to the two-component Hunter–Saxton system, which is used by Wunsch [44] to show that there are global conservative solutions. Kohlmann [25] further elaborates on the geometric interpretation of the two-component Hunter–Saxton system. Finally, Wunsch [43] proved that there are global dissipative solutions to the two-component Hunter–Saxton system on $\mathbb{R}$.

If $(\sigma, A) = (1, 0)$ and $\rho \equiv 0$, then (1.1) reduces to the Hunter–Saxton equation [17],

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0. \tag{1.3}$$

In the Hunter–Saxton equation [17], x is the space variable in a reference frame moving with the linearized wave velocity, t is a slow-time variable, and $u(t,x)$ is a measure of the average orientation of the medium locally around x at time t. More precisely, the orientation of the molecules is described by the field of unit vectors $(\cos u(t,x), \sin u(t,x))$ [45]. The Hunter–Saxton equation also arises in a different physical context as the high-frequency limit [12,18] of the Camassa–Holm equation for shallow water waves [4,21] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [11] with a bi-Hamiltonian structure [17,38] which is completely integrable [1,18]. The initial value problem for the Hunter–Saxton equation on the line (nonperiodic case) and on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ were studied by Hunter and Saxton [17] using the method of characteristics and by Yin [45] using the Kato semigroup method. Moreover, the two classes of admissible weak solutions, dissipative and conservative solutions, and their stability were studied in [2,3,19]. Lenells [28] verified that the Hunter–Saxton equation also describes the geodesic flows on the quotient space of the infinite-dimensional group $D^s(\mathbb{S})$ modulo the subgroup of rotations $Rot(\mathbb{S})$.

Recently, the authors of [41] have studied the global existence of solutions to a two-component generalized Hunter–Saxton system in the periodic setting for the particular choice of the parameter $\sigma = 1$. The aim of this paper is to study the wave breaking and global existence for the generalized periodic two-component Hunter–Saxton system for the parameter $\sigma \in \mathbb{R}$ and to determine a wave-breaking criterion for strong solutions by using the localization analysis in the transport equation theory.

Our main results of this paper are Theorems 3.1–3.2 (wave-breaking criterion), Theorems 4.1–4.2 (wave-breaking data), Theorem 4.3 (blow-up rate), and Theorems 5.1–5.2 (global solution).

The remainder of the paper is organized as follows. In Section 2, the local well-posedness for (1.1) with the initial data in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geqslant 2$, is established. Section 3 deals with the wave breaking of this new system. Using transport equation theory, Theorem 3.1 states a wave-breaking criterion which says that the wave breaking only depends on the slope of u, not the slope of ρ. Theorem 3.2 improves the blow-up criterion with a more precise condition. In Section 4, there are various detailed results of wave breaking and blow-up rate of strong solutions. Finally, Section 5 provides a sufficient condition for global solutions.

Notations. Throughout this paper, $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ shall denote the unit circle. By $H^s(\mathbb{S})$, $s \geqslant 0$, we will represent the Sobolev spaces of equivalence classes of functions defined on the unit circle $\mathbb{S}$ which have square-integrable distributional derivatives up to order s. The $H^s(\mathbb{S})$-norm will be designated by $\|\cdot\|_{H^s}$ and the norm of a vector $v \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ will be written as $\|v\|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})}$. Also, the Lebesgue spaces of order $p \in [1,\infty]$ will be denoted by $L^p(\mathbb{S})$, and the norms of their elements by $\|f\|_{L^p(\mathbb{S})}$. Finally, if $p = 2$, we agree on the convention $\|\cdot\|_{L^2(\mathbb{S})} := \|\cdot\|$.

2. Preliminaries

In this section, we first present the derivation of our model which is the short-wave limit of the generalized two-component Camassa–Holm system (gCH2) [5]. Then we will apply Kato's theory to establish the local well-posedness for the Cauchy problem of system (1.1) and we briefly give the needed results to pursue our goal.

Most recently, the generalized two-component Camassa–Holm system (gCH2)

$$\begin{cases} u_t - u_{txx} - 2\sigma u_x u_{xx} - \sigma u u_{xxx} + \rho\rho_x - Au_x + 3uu_x = 0, \\ \rho_t + (\rho u)_x = 0 \end{cases} \tag{2.1}$$

was derived in [5] by using Ivanov's modeling approach [20]. Here we consider the short-wave limit of the gCH2 equation. Let

$$\tau = \epsilon t, \qquad \zeta = \epsilon^{-1} x, \tag{2.2}$$

and expand u and ρ in power series as follows,

$$u = \epsilon^2\big(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots\big), \tag{2.3}$$

$$\rho = \epsilon\big(\rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots\big). \tag{2.4}$$

Then we have

$$\frac{\partial}{\partial t} = \epsilon\frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = \epsilon^{-1}\frac{\partial}{\partial \zeta}. \tag{2.5}$$

Substituting this relation (2.5) into Eq. (2.1), we get

$$\begin{cases} \epsilon u_\tau - \epsilon^{-1} u_{\tau\zeta\zeta} - 2\sigma\epsilon^{-3} u_\zeta u_{\zeta\zeta} - \sigma\epsilon^{-3} u u_{\zeta\zeta\zeta} + \epsilon^{-1}\rho\rho_\zeta - A\epsilon^{-1} u_\zeta + 3\epsilon^{-1} u u_\zeta = 0, \\ \epsilon\rho_\tau + \epsilon^{-1}(\rho u)_\zeta = 0. \end{cases} \tag{2.6}$$

Applying the series (2.3), (2.4) to Eq. (2.6), we obtain the following partial differential equation for u_0 and ρ_0 at the lowest order in ϵ,

$$\begin{cases} -u_{0\tau\zeta\zeta} - 2\sigma u_{0\zeta} u_{0\zeta\zeta} - \sigma u u_{0\zeta\zeta\zeta} + \rho\rho_{0\zeta} - Au_{0\zeta} = 0, \\ \rho_{0\tau} + (\rho_0 u_0)_\zeta = 0. \end{cases} \tag{2.7}$$

Writing the above equations (2.7) in terms of the original variables t and x, we obtain the following generalized two-component Hunter–Saxton system (2.8),

$$\begin{cases} u_{txx} + 2\sigma u_x u_{xx} + \sigma u u_{xxx} - \rho\rho_x + Au_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{2.8}$$

We now provide the framework in which we shall reformulate (1.1). In order to do this, we observe that we can write the first equation of (1.1) in the integrated form

$$u_{tx} + \frac{\sigma}{2}u_x^2 + \sigma u u_{xx} - \frac{1}{2}\rho^2 + Au = g(t), \tag{2.9}$$

where $g(t)$ is determined by the periodicity of u to be

$$g(t) = -\int_{\mathbb{S}} \left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au\right) dx. \tag{2.10}$$

Integrating both sides of (2.9) with respect to variable x, we obtain

$$u_t+\sigma uu_x=\partial_x^{-1}\left(\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2-Au+g\right)+h(t), \tag{2.11}$$

where $\partial_x^{-1}f(x):=\int_0^x f(y)\,dy$ and $h(t):[0,\infty)\to\mathbb{R}$ is an arbitrary continuous function. Thus (1.1) can be written in the following "transport" form

$$\begin{cases} u_t+\sigma uu_x=\partial_x^{-1}\left(\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2-Au+g\right)+h(t), & t>0,\ x\in\mathbb{R},\\ \rho_t+u\rho_x=-u_x\rho, \quad t>0,\ x\in\mathbb{R}, \\ u(t,x+1)=u(t,x), \quad \rho(t,x+1)=\rho(t,x), \quad t\geqslant 0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x), \quad \rho(0,x)=\rho_0(x), \quad x\in\mathbb{R}, \end{cases} \tag{2.12}$$

where $\partial_x^{-1}f(x):=\int_0^x f(y)\,dy$ and $h(t):[0,\infty)\to\mathbb{R}$ is an arbitrary continuous function.

Next, we will apply Kato's theory to establish the local well-posedness for system (1.1). For convenience, we state here Kato's theory in the form suitable for our purpose. Consider the abstract quasi-linear evolution equation

$$\frac{dv}{dt}+A(v)v=f(v), \quad t\geqslant 0, \quad v(0)=v_0. \tag{2.13}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X and let $Q:Y\to X$ be a topological isomorphism. Let $L(Y,X)$ denote the space of all bounded linear operators from Y to X (if $X=Y$, $L(Y,X)=L(X)$). We assume that:

(i) $A(y)\in L(Y,X)$ for $y\in X$ with

$$\big\|\big[A(y)-A(z)\big]w\big\|_X\leqslant\mu_1\|y-z\|_X\|w\|_Y, \quad y,z,w\in Y,$$

and $A(y)$ is quasi-m-accretive uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1}=A(y)+B(y)$, where $B(y)\in L(X)$ is bounded uniformly on bounded sets in Y. Moreover,

$$\big\|\big[B(y)-B(z)\big]w\big\|_X\leqslant\mu_2\|y-z\|_Y\|w\|_X, \quad y,z\in Y,\ w\in X.$$

(iii) $f:Y\to Y$ extends to a map from X into X. f is bounded on bounded sets in Y, and

$$\big\|f(y)-f(z)\big\|_Y\leqslant\mu_3\|y-z\|_Y, \quad y,z\in Y,$$
$$\big\|f(y)-f(z)\big\|_X\leqslant\mu_4\|y-z\|_X, \quad y,z\in Y.$$

Here $(\mu_i)_{i=1}^4$ depends only on $\max\{\|y\|_Y,\|z\|_Y\}$. With these conditions, we can state Kato's theorem.

Proposition 2.1. *(See [23].) Given the evolution equation* (2.13), *assume that the conditions* (i), (ii), *and* (iii) *hold. For a fixed $v_0\in Y$, there is a maximal $T>0$ depending only on $\|v_0\|_Y$ and a unique solution v to the abstract quasi-linear evolution equation* (2.13) *such that*

$$v=v(\cdot,v_0)\in C\big([0,T);Y\big)\cap C^1\big([0,T);X\big).$$

Moreover, the map $v_0\to v(\cdot,v_0)$ is continuous from Y to

$$C\big([0,T);Y\big)\cap C^1\big([0,T);X\big).$$

In order to make Kato's theorem more applicable to (1.1), write $z = \binom{u}{\rho}$. Define $A(z) := \begin{pmatrix} \sigma u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}$, $f(z) := \binom{\partial_x^{-1}(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g(t)) + h(t)}{-\rho u_x}$, so that (1.1) becomes the abstract evolution equation

$$\begin{cases} \dfrac{dz}{dt} + Az = f(z), \\ z(0,x) = z_0(x) = \begin{pmatrix} u_0(x) \\ \rho_0(x) \end{pmatrix}. \end{cases} \tag{2.14}$$

Set $Y = H^s \times H^{s-1}$, $X = H^{s-1} \times H^{s-2}$, $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$ and $Q = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$. Obviously, Q is an isomorphism of $H^s \times H^{s-1}$ onto $H^{s-1} \times H^{s-2}$. In order to prove the local well-posedness for system (1.1), we only need to verify $A(z)$ and $f(z)$ satisfy the conditions (i)–(iii).

Theorem 2.1. *Given any $X_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geqslant 2$, there exist a maximal $T = T(\sigma, A; \|X_0\|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})}) > 0$, and a unique solution $X = \binom{u}{\rho}$ to (1.1) such that*

$$X = X(\cdot, X_0) \in C\big([0,T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})\big).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping $X_0 \mapsto X(\cdot, X_0) : H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \to C([0,T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$ is continuous and the maximal existence time T can be chosen independently of the Sobolev order s.

The following lemma will facilitate the required computations.

Lemma 2.1. *(See [22,45].) Let r and t be real numbers such that $-r < t \leqslant r$. Then*

$$\|fg\|_{H^t} \leqslant c\|f\|_{H^r}\|g\|_{H^t}, \quad \text{if } r > \frac{1}{2},$$

where c is a positive constant independent of f and g.

Proof of Theorem 2.1. If $\sigma = 1$, then conditions (i) and (ii) have been shown to hold for

$$A(z) = \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix} \in L\big(H^s \times H^{s-1}, H^{s-1} \times H^{s-2}\big)$$

in [14]. Since the other cases are dealt with in the same fashion, we omit the proof of conditions (i) and (ii) and only verify condition (iii).

For any two vectors $z = \binom{u}{\rho}$, $y = \binom{v}{\mu} \in H^s \times H^{s-1}$,

$$\begin{aligned}
&\big\|f(y) - f(z)\big\|_{H^s \times H^{s-1}} \\
&\quad\leqslant \left\|\partial_x^{-1}\left[\frac{\sigma}{2}\big(v_x^2 - u_x^2\big) + \frac{1}{2}\big(\mu^2 - \rho^2\big) - A(v-u)\right]\right\|_{H^s} + \|\mu v_x - \rho u_x\|_{H^{s-1}} \\
&\quad\leqslant \frac{|\sigma|}{2}\big\|\partial_x^{-1}\big(v_x^2 - u_x^2\big)\big\|_{H^s} + \frac{1}{2}\big\|\partial_x^{-1}\big(\mu^2 - \rho^2\big)\big\|_{H^s} + A\big\|\partial_x^{-1}(v-u)\big\|_{H^s} + \|\mu v_x - \rho u_x\|_{H^{s-1}} \\
&\quad\leqslant \frac{|\sigma|}{2}\big\|v_x^2 - u_x^2\big\|_{H^{s-1}} + \frac{1}{2}\big\|\mu^2 - \rho^2\big\|_{H^{s-1}} + A\|v-u\|_{H^{s-1}} \\
&\qquad + \big\|\mu(v-u)_x\big\|_{H^{s-1}} + \big\|(\mu-\rho)u_x\big\|_{H^{s-1}}
\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{|\sigma|}{2}\|v+u\|_{H^s}\|v-u\|_{H^s}+\frac{1}{2}\|\mu+\rho\|_{H^{s-1}}\|\mu-\rho\|_{H^{s-1}}\\
&\quad + A\|v-u\|_{H^{s-1}}+\|\mu\|_{H^{s-1}}\|v-u\|_{H^s}+\|u\|_{H^s}\|\mu-\rho\|_{H^{s-1}}\\
&\leqslant \mu_3\|y-z\|_{H^s\times H^{s-1}},
\end{aligned}$$

where the constant μ_3 depends only on σ, A, $\|y\|_{H^s\times H^{s-1}}$, $\|z\|_{H^s\times H^{s-1}}$. Taking $y=0$ in the above inequality, we obtain that f is bounded on bounded set in $H^s\times H^{s-1}$.

For the last estimate, we similarly compute

$$\begin{aligned}
&\big\|f(y)-f(z)\big\|_{H^{s-1}\times H^{s-2}}\\
&\leqslant \frac{|\sigma|}{2}\big\|\partial_x^{-1}\big(v_x^2-u_x^2\big)\big\|_{H^{s-1}}+\frac{1}{2}\big\|\partial_x^{-1}\big(\mu^2-\rho^2\big)\big\|_{H^{s-1}}+A\big\|\partial_x^{-1}(v-u)\big\|_{H^{s-1}}+\|\mu v_x-\rho u_x\|_{H^{s-2}}\\
&\leqslant \frac{|\sigma|}{2}\big\|v_x^2-u_x^2\big\|_{H^{s-2}}+\frac{1}{2}\big\|\mu^2-\rho^2\big\|_{H^{s-2}}+A\|v-u\|_{H^{s-2}}\\
&\quad+\big\|\mu(v-u)_x\big\|_{H^{s-2}}+\big\|(\mu-\rho)u_x\big\|_{H^{s-2}}\\
&\leqslant \frac{|\sigma|}{2}\|v+u\|_{H^{s-1}}\|v-u\|_{H^{s-1}}+\frac{1}{2}\|\mu+\rho\|_{H^{s-2}}\|\mu-\rho\|_{H^{s-2}}\\
&\quad+A\|v-u\|_{H^{s-2}}+\|\mu\|_{H^{s-2}}\|v-u\|_{H^{s-1}}+\|u\|_{H^{s-1}}\|\mu-\rho\|_{H^{s-2}}\\
&\leqslant \mu_4\|y-z\|_{H^{s-1}\times H^{s-2}},
\end{aligned}$$

where $\mu_4=\mu_4(\sigma, A, \|y\|_{H^{s-1}\times H^{s-2}}, \|z\|_{H^{s-1}\times H^{s-2}})$. Thus the proof is complete. □

Given any initial data $X_0=\binom{u_0}{\rho_0}\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ with $s\geqslant 2$, Theorem 2.1 ensures the existence of a maximal $T=T(\sigma, A; \|X_0\|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})})>0$ and a unique solution $X=\binom{u}{\rho}$ to (1.1) such that

$$X=X(\cdot, X_0)\in C\big([0,T); H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})\big)\cap C^1\big([0,T); H^{s-1}(\mathbb{S})\times H^{s-2}(\mathbb{S})\big).$$

Now, consider the initial value problem for the Lagrangian flow map:

$$\begin{cases}\dfrac{\partial\varphi}{\partial t}=u\big(t,\varphi(t,x)\big), & t\in[0,T),\\ \varphi(0,x)=x, & x\in\mathbb{R},\end{cases} \tag{2.15}$$

where u denotes the first component of the solution X to (1.1). Applying classical results from ordinary differential equations, one can obtain the following result on φ which is crucial in the proof of the blow-up scenarios.

Lemma 2.2. *Let $u\in C([0,T); H^s)\cap C^1([0,T); H^{s-1})$, $s\geqslant 2$. Then initial value problem* (2.15) *admits a unique solution $\varphi\in C^1([0,T)\times\mathbb{R};\mathbb{R})$. Moreover, $\{\varphi(t,\cdot)\}_{t\in[0,T)}$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$\varphi_x(t,x)=e^{\int_0^t u_x(\tau,\varphi(\tau,x))\,d\tau}>0, \quad (t,x)\in[0,T)\times\mathbb{R}. \tag{2.16}$$

Proof. Since $u\in C^1([0,T); H^{s-1})$ and $H^s(\mathbb{S})\hookrightarrow C^1(\mathbb{S})$, we see that both functions $u(t,x)$ and $u_x(t,x)$ are bounded and Lipschitz in the space variable x, and of class C^1 in time. Therefore, for fixed $x\in\mathbb{R}$, (2.15) is an ordinary differential equation. Then well-known classical results from ordinary differential equation tell us that (2.15) has a unique solution $\varphi(t,x)\in C^1([0,T)\times\mathbb{R};\mathbb{R})$.

Differentiation of (2.15) with respect to x yields

$$\begin{cases} \dfrac{d}{dt}\varphi_x = u_x\big(t, \varphi(t,x)\big)\varphi_x, & t \in [0, T), \\ \varphi_x(0, x) = 1, & x \in \mathbb{R}. \end{cases} \tag{2.17}$$

The solution to (2.17) is given by

$$\varphi_x(t, x) = e^{\int_0^t u_x(\tau, \varphi(\tau, x))\, d\tau}, \quad (t, x) \in [0, T) \times \mathbb{R}. \tag{2.18}$$

For every $T_0 < T$, it follows from the Sobolev embedding theorem that

$$\sup_{(\tau,x)\in[0,T_0)\times\mathbb{R}} \big|u_x(\tau, x)\big| < \infty.$$

We infer from (2.18) that there exists a constant $M > 0$ such that

$$\varphi_x(t, x) \geqslant e^{-Mt}, \quad (t, x) \in [0, T) \times \mathbb{R},$$

which implies that the map $\varphi(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$\varphi_x(t, x) = e^{\int_0^t u_x(\tau, \varphi(\tau, x))\, d\tau} > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.$$

This completes the proof of Lemma 2.2. $\square$

Remark 2.1. Since $\varphi(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line for every $t \in [0, T)$, the L^∞-norm of any function $v(t, \cdot) \in L^\infty$, $t \in [0, T)$, is preserved under the family of diffeomorphisms $\varphi(t, \cdot)$ with $t \in [0, T)$, that is,

$$\big\|v(t, \cdot)\big\|_{L^\infty(\mathbb{S})} = \big\|v\big(t, \varphi(t, \cdot)\big)\big\|_{L^\infty(\mathbb{S})}, \quad t \in [0, T).$$

Similarly, we have

$$\begin{aligned} \inf_{x\in\mathbb{S}} v(t, x) &= \inf_{x\in\mathbb{S}} v\big(t, \varphi(t, x)\big), \quad t \in [0, T), \\ \sup_{x\in\mathbb{S}} v(t, x) &= \sup_{x\in\mathbb{S}} v\big(t, \varphi(t, x)\big), \quad t \in [0, T). \end{aligned}$$

Lemma 2.3. *Suppose that $\sigma \in \mathbb{R}$. Let $X = \binom{u}{\rho}$ be a smooth solution to* (1.1)*. Then*

$$\frac{d}{dt} g(t) = \frac{(1-\sigma)}{2} \int_{\mathbb{S}} u_x \rho^2\, dx + A \int_{\mathbb{S}} \partial_x^{-1}\left(\frac{\sigma}{2} u_x^2 + \frac{1}{2}\rho^2 - Au + g\right) dx + Ah(t). \tag{2.19}$$

Proof. By using (2.9)–(2.11), we have

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt}\left(-\int_{\mathbb{S}} \left(\frac{\sigma}{2} u_x^2 + \frac{1}{2}\rho^2 - Au\right) dx\right) \\ &= -\sigma \int_{\mathbb{S}} u_x u_{tx}\, dx - \int_{\mathbb{S}} \rho\rho_t\, dx + A \int_{\mathbb{S}} u_t\, dx \end{aligned}$$

$$
\begin{aligned}
&= -\sigma \int_{\mathbb{S}} u_x\left(-\frac{\sigma}{2}u_x^2 - \sigma u u_{xx} + \frac{1}{2}\rho^2 - Au + g\right)dx + \int_{\mathbb{S}} \rho(\rho u)_x\,dx \\
&\quad + A\int_{\mathbb{S}}\left[-\sigma u u_x + \partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right) + h(t)\right]dx \\
&= \frac{\sigma^2}{2}\int_{\mathbb{S}} u_x^3\,dx + \sigma^2\int_{\mathbb{S}} u u_x u_{xx}\,dx - \frac{\sigma}{2}\int_{\mathbb{S}} u_x\rho^2\,dx + \sigma A\int_{\mathbb{S}} u u_x\,dx - \sigma g(t)\int_{\mathbb{S}} u_x\,dx \\
&\quad + \int_{\mathbb{S}} \rho(\rho u)_x\,dx - \sigma A\int_{\mathbb{S}} u u_x\,dx + A\int_{\mathbb{S}} \partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right)dx + Ah(t) \\
&= -\frac{\sigma}{2}\int_{\mathbb{S}} u_x\rho^2\,dx - \int_{\mathbb{S}} \rho_x\rho u\,dx + A\int_{\mathbb{S}} \partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right)dx + Ah(t) \\
&= \frac{(1-\sigma)}{2}\int_{\mathbb{S}} u_x\rho^2\,dx + A\int_{\mathbb{S}} \partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right)dx + Ah(t). \qquad \square
\end{aligned}
$$

Remark 2.2. In particular, if $(\sigma, A) = (1, 0)$, then $\frac{d}{dt}g(t) = 0$, which implies that the system enjoys a conservation law, namely,

$$
g(t) \equiv g(0) = -\frac{1}{2}\int_{\mathbb{S}} u_{0,x}^2 + \rho_0^2\,dx \tag{2.20}
$$

is constant for all $t \geqslant 0$.

Lemma 2.4. *Let $X_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geqslant 2$, and let T be the maximal existence time of the solution $X = \binom{u}{\rho}$ to* (1.1) *with initial data X_0. Then for all $t \in [0, T)$, we have the following conservation laws*

$$
\int_{\mathbb{S}} \rho(t,x)\,dx = \int_{\mathbb{S}} \rho_0(x)\,dx, \tag{2.21}
$$

$$
\int_{\mathbb{S}} u_x^2(t,x) + \rho^2(t,x)\,dx = \int_{\mathbb{S}} u_{0,x}^2(x) + \rho_0^2(x)\,dx. \tag{2.22}
$$

Proof. Integrating the second equation in (1.1) by parts, in view of the periodicity of u and ρ, we get

$$
\frac{d}{dt}\int_{\mathbb{S}} \rho\,dx = -\int_{\mathbb{S}} (u\rho)_x\,dx = 0.
$$

On the other hand, multiplying the equation in (2.9) by u_x and integrating by parts, in view of the periodicity of u, we get

$$
\frac{d}{dt}\int_{\mathbb{S}} u_x^2\,dx = -2\int_{\mathbb{S}} u\rho\rho_x\,dx.
$$

Multiplying the second equation in (1.1) by ρ and integrating by parts, we have

$$\frac{d}{dt}\int_{\mathbb{S}} \rho^2\,dx = 2\int_{\mathbb{S}} u\rho\rho_x\,dx.$$

Adding the above two equations, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^2+\rho^2\,dx = 0.$$

This completes the proof of Lemma 2.4. □

For the sake of convenience, let

$$E_0=\int_{\mathbb{S}} u_x^2(t,x)+\rho^2(t,x)\,dx=\int_{\mathbb{S}} u_{0,x}^2(x)+\rho_0^2(x)\,dx, \tag{2.23}$$

$$E_1=\int \rho(t,x)\,dx=\int \rho_0(x)\,dx. \tag{2.24}$$

Then E_0 and E_1 are constants and independent of time t.

Lemma 2.5. *Let $X_0=\binom{u_0}{\rho_0}\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s\geqslant 2$, and let T be the maximal existence time of the solution $X=\binom{u}{\rho}$ to (1.1) with initial data X_0. Then we have*

$$\int_{\mathbb{S}} u^2(t,x)\,dx\leqslant e^{C_2 t}\biggl(\int_{\mathbb{S}} u_0^2(x)\,dx+1\biggr),\quad \forall t\in[0,T), \tag{2.25}$$

where $C_1=\max(|\sigma|,1)E_0+\sup_{t\in[0,\infty)}|h(t)|>0$, $C_2=C_1+4A$.

Proof. By direct computation with conservation law E_0, we have

$$\begin{aligned}
&\left|\partial_x^{-1}\left(\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2-Au+g\right)+h(t)\right|\\
&\quad\leqslant\int_0^1\left|\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2-Au+g\right|dx+\bigl|h(t)\bigr|\\
&\quad\leqslant\frac{1}{2}\max\bigl(|\sigma|,1\bigr)E_0+\bigl|g(t)\bigr|+\bigl|h(t)\bigr|+A\int_0^1|u|\,dx\\
&\quad\leqslant\max\bigl(|\sigma|,1\bigr)E_0+\bigl|h(t)\bigr|+2A\int_{\mathbb{S}}|u|\,dx\\
&\quad\leqslant\max\bigl(|\sigma|,1\bigr)E_0+\sup_{t\in[0,\infty)}\bigl|h(t)\bigr|+2A\int_{\mathbb{S}}|u|\,dx:=C_1+2A\int_{\mathbb{S}}|u|\,dx,
\end{aligned}\tag{2.26}$$

where $C_1 = \max(|\sigma|, 1)E_0 + \sup_{t\in[0,\infty)} |h(t)| > 0$, and

$$\left|g(t)\right| = \left|-\int_{\mathbb{S}} \left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au\right) dx\right| \leqslant \frac{1}{2}\max\bigl(|\sigma|, 1\bigr)E_0 + A\int_{\mathbb{S}} |u|\,dx. \tag{2.27}$$

Multiplying Eq. (2.11) by u and integrating with respect to x, in view of the periodicity of u and (2.26), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} u^2(t,x)\,dx &= \int_{\mathbb{S}} uu_t\,dx \\
&= -\sigma\int_{\mathbb{S}} u_x u^2\,dx + \int_{\mathbb{S}} u\left[\partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right) + h(t)\right]dx \\
&= \int_{\mathbb{S}} u\left[\partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right) + h(t)\right]dx \\
&\leqslant \left(C_1 + 2A\int_{\mathbb{S}} |u|\,dx\right)\int_{\mathbb{S}} |u|\,dx \leqslant C_1\int_{\mathbb{S}} |u|\,dx + 2A\left(\int_{\mathbb{S}} |u|\,dx\right)^2.
\end{aligned} \tag{2.28}$$

By using the Cauchy–Schwarz inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} u^2(t,x)\,dx \leqslant \left(\frac{C_1}{2} + 2A\right)\int_{\mathbb{S}} u^2\,dx + \frac{C_1}{2} := \frac{C_2}{2}\int_{\mathbb{S}} u^2\,dx + \frac{C_1}{2}, \tag{2.29}$$

where $C_2 = C_1 + 4A$; note that $C_2 > C_1$.

By Gronwall's inequality, we get

$$\int_{\mathbb{S}} u^2(t,x)\,dx \leqslant e^{C_2 t}\left(\int_{\mathbb{S}} u_0^2(x)\,dx + \frac{C_1}{C_2}\right) - \frac{C_1}{C_2} \leqslant e^{C_2 t}\left(\int_{\mathbb{S}} u_0^2(x)\,dx + 1\right). \tag{2.30}$$

This completes the proof of Lemma 2.5. □

Lemma 2.6. *Assume that $u_0 \in H^s(\mathbb{S})$, $s \geqslant 2$, $u_0 \not\equiv 0$, and the corresponding solution $u(t,x)$ of* (1.1) *has a zero for any time $t \geqslant 0$. Then for all $t \in [0,T)$ we have*

$$\int_{\mathbb{S}} u^2(t,x)\,dx \leqslant E_0. \tag{2.31}$$

Moreover, if $u(t,x)$ is odd with respect to x, we also have (2.31).

Proof. By assumption, there is $x_0 \in [0,1]$ such that $u(t,x_0) = 0$ for each $t \in [0,T)$. Then for $x \in \mathbb{S}$ we have

$$u^2(t,x) = \left(\int_{x_0}^{x} u_x\,dx\right)^2 \leqslant (x - x_0)\int_{x_0}^{x} u_x^2\,dx, \quad x \in [x_0, x_0 + 1/2].$$

This implies

$$\sup_{x\in\mathbb{S}} u^2(t,x) \leqslant \frac{1}{2}\int_{\mathbb{S}} u_x^2\,dx.$$

Using conservation law E_0, it follows that

$$\int_{\mathbb{S}} u^2(t,x)\,dx \leqslant \sup_{x\in\mathbb{S}} u^2(t,x) \leqslant \frac{1}{2}\int_{\mathbb{S}} u_x^2\,dx \leqslant \int_{\mathbb{S}} u_x^2+\rho^2\,dx = E_0.$$

Since $u(t,x)$ is odd with respect to x, we have $u(t,0)=0$. Thus, if we set $x_0=0$, then we also have (2.31). This completes the proof of Lemma 2.6. $\square$

3. Wave-breaking criteria

In this section, we present the wave-breaking criteria for solutions to (1.1) by using transport equation theory. We first recall the following propositions.

Proposition 3.1 *(1-D Moser-type estimates). (See [15].) The following estimates hold*:

(i) *For* $s \geqslant 0$,

$$\|fg\|_{H^s(\mathbb{R})} \leqslant C\big(\|f\|_{L^\infty(\mathbb{R})}\|g\|_{H^s(\mathbb{R})} + \|f\|_{H^s(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}\big). \tag{3.1}$$

(ii) *For* $s > 0$,

$$\|f\partial_x g\|_{H^s(\mathbb{R})} \leqslant C\big(\|f\|_{L^\infty(\mathbb{R})}\|\partial_x g\|_{H^s(\mathbb{R})} + \|f\|_{H^{s+1}(\mathbb{R})}\|g\|_{L^\infty(\mathbb{R})}\big). \tag{3.2}$$

(iii) *For* $s_1 \leqslant \frac{1}{2}$, $s_2 > \frac{1}{2}$ *and* $s_1+s_2>0$,

$$\|fg\|_{H^{s_1}(\mathbb{R})} \leqslant C\|f\|_{H^{s_1}(\mathbb{R})}\|g\|_{H^{s_2}(\mathbb{R})}, \tag{3.3}$$

where C's are constants independent of f and g.

Proposition 3.2. *(See [15].) Suppose that* $s > -\frac{d}{2}$. *Let* v *be a vector field such that* ∇v *belongs to* $L^1([0,T];H^{s-1})$ *if* $s > 1+\frac{d}{2}$ *or to* $L^1([0,T];H^{\frac{d}{2}}\cap L^\infty)$ *otherwise. Suppose also that* $f_0 \in H^s$, $F \in L^1([0,T];H^s)$ *and that* $f \in L^\infty([0,T];H^s)\cap C([0,T];S')$ *solves the d-dimensional linear transport equations*

$$(T)\qquad \begin{cases} \partial_t f + v\cdot\nabla f = F, \\ f|_{t=0} = f_0. \end{cases}$$

Then $f \in C([0,T];H^s)$. *More precisely, there exists a constant C depending only on s, p and d such that the following statements hold*:

(1) *If* $s \neq 1+\frac{d}{2}$,

$$\|f\|_{H^s} \leqslant \|f_0\|_{H^s} + \int_0^t \big\|F(\tau)\big\|_{H^s}\,d\tau + C\int_0^t V'(\tau)\big\|f(\tau)\big\|_{H^s}\,d\tau, \tag{3.4}$$

or

$$\|f\|_{H^s} \leqslant e^{CV(t)}\left(\|f_0\|_{H^s} + \int_0^t e^{-CV(\tau)}\big\|F(\tau)\big\|_{H^s}\,d\tau\right), \tag{3.5}$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{\frac{d}{2}} \cap L^\infty}\,d\tau$ *if* $s < 1 + \frac{d}{2}$ *and* $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{s-1}}\,d\tau$ *else.*

(2) *If* $f = v$, *then for all* $s > 0$, *the estimates* (3.4) *and* (3.5) *hold with* $V(t) = \int_0^t \|\partial_x u(\tau)\|_{L^\infty} d\tau$.

Proposition 3.3. *(See [15].) Let* $0 < s < 1$. *Suppose that* $f_0 \in H^s$, $g \in L^1([0,T]; H^s)$, $v, \partial_x v \in L^1([0,T]; L^\infty)$ *and that* $f \in L^\infty([0,T]; H^s) \cap C([0,T]; S')$ *solves the* 1*-dimensional linear transport equation*

$$(T) \quad \begin{cases} \partial_t f + v \cdot \partial_x f = g, \\ f|_{t=0} = f_0. \end{cases}$$

Then $f \in C([0,T]; H^s)$. *More precisely, there exists a constant* C *depending only on* s *such that the following statements hold*:

$$\|f\|_{H^s} \leqslant \|f_0\|_{H^s} + C\int_0^t \big\|g(\tau)\big\|_{H^s}\,d\tau + C\int_0^t \big\|f(\tau)\big\|_{H^s} V'(\tau)\,d\tau, \tag{3.6}$$

or

$$\|f\|_{H^s} \leqslant e^{CV(t)}\left(\|f_0\|_{H^s} + C\int_0^t \big\|g(\tau)\big\|_{H^s}\,d\tau\right), \tag{3.7}$$

with $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{L^\infty})\,d\tau$.

The above proposition was proved in [38] using Littlewood–Paley analysis for the transport equation and Moser-type estimates. Using this result and performing the same argument, we can obtain the following blow-up criterion.

Theorem 3.1. *Let* $X_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ *with* $s \geqslant 2$, *and* $X = \binom{u}{\rho}$ *be the corresponding solution to* (1.1)*. Assume* $T > 0$ *is the maximal time of existence. Then*

$$T < \infty \quad \Longrightarrow \quad \int_0^T \big\|\partial_x u(\tau)\big\|_{L^\infty}\,d\tau = \infty. \tag{3.8}$$

Proof. We shall prove this theorem by an inductive argument with respect to the index s. To this end, let us first give a control on $\|\rho\|_{L^\infty}$ and $\|u\|_{L^\infty}$.

In fact, applying the maximal principle to the transport equation about ρ,

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{3.9}$$

we have

$$\left\|\rho(t)\right\|_{L^\infty} \leqslant \|\rho_0\|_{L^\infty} + C\int_0^t \|\partial_x u\|_{L^\infty}\|\rho\|_{L^\infty}\,d\tau.$$

A simple application of Gronwall's inequality implies

$$\left\|\rho(t)\right\|_{L^\infty} \leqslant \|\rho_0\|_{L^\infty} e^{C\int_0^t \|\partial_x u\|_{L^\infty}\,d\tau}. \tag{3.10}$$

Now let us concentrate our attention to the proof of Theorem 3.1. This can be achieved as follows:

Step 1. For $2 < s < 3$, applying Proposition 3.3 to the transport equation with respect to ρ,

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{3.11}$$

we have

$$\left\|\rho(t)\right\|_{H^{s-2}} \leqslant \|\rho_0\|_{H^{s-2}} + C\int_0^t \|\rho\partial_x u\|_{H^{s-2}}\,d\tau + C\int_0^t \|\rho\|_{H^{s-2}}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau.$$

Using (3.1), one has

$$\|\rho\partial_x u\|_{H^{s-2}} \leqslant C\big(\|\rho\|_{H^{s-2}}\|\partial_x u\|_{L^\infty} + \|\partial_x u\|_{H^{s-2}}\|\rho\|_{L^\infty}\big). \tag{3.12}$$

Therefore, we have

$$\begin{aligned}\left\|\rho(t)\right\|_{H^{s-2}} &\leqslant \|\rho_0\|_{H^{s-2}} + C\int_0^t \left\|\partial_x u(\tau)\right\|_{H^{s-2}}\left\|\rho(\tau)\right\|_{L^\infty}\,d\tau \\ &\quad + C\int_0^t \left\|\rho(\tau)\right\|_{H^{s-2}}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau.\end{aligned} \tag{3.13}$$

By differentiating (3.11) once with respect to x, we have

$$\partial_t\rho_x + u\partial_x(\rho_x) + 2u_x\rho_x + \rho u_{xx} = 0. \tag{3.14}$$

Proposition 3.3 applied to (3.14) implies that

$$\begin{aligned}\left\|\rho_x(t)\right\|_{H^{s-2}} &\leqslant \|\rho_{0,x}\|_{H^{s-2}} + C\int_0^t \left\|(2u_x\rho_x + \rho\partial_x u_x)(\tau)\right\|_{H^{s-2}}\,d\tau \\ &\quad + C\int_0^t \left\|\rho_x(\tau)\right\|_{H^{s-2}}\big(\left\|u(\tau)\right\|_{L^\infty} + \left\|\partial_x u(\tau)\right\|_{L^\infty}\big)\,d\tau\end{aligned}$$

$$\leqslant \|\rho_{0,x}\|_{H^{s-2}} + C\int_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}}\big)\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty} + \|\rho\|_{L^\infty}\big)\,d\tau, \qquad (3.15)$$

where we used (3.2):

$$\|u_x\rho_x\| \leqslant C\big(\|\partial_x u\|_{H^{s-1}}\|\rho\|_{L^\infty} + \|\partial_x\rho\|_{H^{s-1}}\|u_x\|_{L^\infty}\big)$$

and

$$\|\rho\,\partial_x u_x\| \leqslant C\big(\|\rho\|_{H^{s-1}}\|\partial_x u\|_{L^\infty} + \|u_{xx}\|_{H^{s-2}}\|\rho\|_{L^\infty}\big).$$

On the other hand, Proposition 3.2 applied to the equation about u,

$$u_t + \sigma u u_x = \partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right) + h(t),$$

implies (for every $s > 1$)

$$\begin{aligned}\big\|u(t)\big\|_{H^s} \leqslant \|u_0\|_{H^s} &+ C\int_0^t \left\|\left[\partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right) + h\right](\tau)\right\|_{H^s} d\tau \\ &+ C\int_0^t \big\|u(\tau)\big\|_{H^s}\big\|\partial_x u(\tau)\big\|_{L^\infty}\,d\tau.\end{aligned}$$

Using (3.1), one has

$$\begin{aligned}&\left\|\partial_x^{-1}\left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right) + h(\tau)\right\|_{H^s} \\ &\quad\leqslant C\left\|\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g\right\|_{H^{s-1}} + \big\|h(\tau)\big\|_{H^s} \\ &\quad\leqslant C\big(\|\partial_x u\|_{H^{s-1}}\|\partial_x u\|_{L^\infty} + \|\rho\|_{H^{s-1}}\|\rho\|_{L^\infty} + \|u\|_{H^{s-1}} + \big|g(\tau)\big|\big) + \max_{\tau\in[0,T)}\big|h(\tau)\big| \\ &\quad\leqslant C\Big(\|\partial_x u\|_{H^{s-1}}\|\partial_x u\|_{L^\infty} + \|\rho\|_{H^{s-1}}\|\rho\|_{L^\infty} + \|u\|_{H^{s-1}} + \big|g(\tau)\big| + \max_{\tau\in[0,T)}\big|h(\tau)\big|\Big).\end{aligned}$$

From this, we obtain

$$\begin{aligned}\big\|u(t)\big\|_{H^s} \leqslant \|u_0\|_{H^s} &+ C\int_0^t \big\|u(\tau)\big\|_{H^s}\big(\big\|\partial_x u(\tau)\big\|_{L^\infty} + 1\big)\,d\tau \\ &+ C\int_0^t \big\|\rho(\tau)\big\|_{H^{s-1}}\big\|\rho(\tau)\big\|_{L^\infty}\,d\tau + C\int_0^t \Big(\big|g(\tau)\big| + \max_{\tau\in[0,T)}\big|h(\tau)\big|\Big)\,d\tau,\end{aligned} \qquad (3.16)$$

which together with (3.13) and (3.15) ensures that

$$\begin{aligned}
&\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \\
&\quad \leqslant \|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + C\int_0^t \Big(|g(\tau)| + \max_{\tau\in[0,T)}|h(\tau)|\Big)\,d\tau \\
&\qquad + C\int_0^t \big(\|u(\tau)\|_{H^s} + \|\rho(\tau)\|_{H^{s-1}}\big)\big(\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty} + \|\rho(\tau)\|_{L^\infty} + 1\big)\,d\tau.
\end{aligned} \tag{3.17}$$

Using Lemma 2.5, we can compute

$$\begin{aligned}
\left|\int_0^t \Big(|g(\tau)| + \max_{\tau\in[0,T)}|h(\tau)|\Big)\,d\tau\right| &\leqslant \Big(|g(t)| + \max_{\tau\in[0,T)}|h(t)|\Big)t \\
&\leqslant \left\{C_1 + A\left(1 + \int_{\mathbb{S}} u^2\,dx\right)\right\}t \\
&\leqslant \left\{C_1 + A + Ae^{C_2 t}\left(1 + \int_{\mathbb{S}} u_0^2(x)\,dx\right)\right\}t \\
&\leqslant \left\{C_1 + A + Ae^{C_2 T}\left(1 + \int_{\mathbb{S}} u_0^2(x)\,dx\right)\right\}T,
\end{aligned}$$

where C_1 and C_2 are given in Lemma 2.5.

Set $K = K(C_1, C_2, A, \|u_0\|, T) := \{C_1 + A + Ae^{C_2 T}(1 + \int_{\mathbb{S}} u_0^2(x)\,dx)\}T$. Using Gronwall's inequality, one can see

$$\begin{aligned}
&\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \\
&\quad \leqslant \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + K\big)e^{C\int_0^t (\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty} + \|\rho(\tau)\|_{L^\infty} + 1)\,d\tau}.
\end{aligned} \tag{3.18}$$

Using the Sobolev embedding theorem $H^s \hookrightarrow L^\infty$ (for $s > \frac{1}{2}$), we get from (2.23) and (2.25) that

$$\begin{aligned}
\|u(t)\|_{L^\infty} &\leqslant C\|u\|_{H^1} \leqslant C\left(\int_{\mathbb{S}} u^2 + u_x^2 + \rho^2\,dx\right)^{\frac{1}{2}} \\
&\leqslant C\left(\int_{\mathbb{S}} u^2\,dx + E_0\right)^{\frac{1}{2}} \leqslant C\left(e^{C_2 t}\left(1 + \int_{\mathbb{S}} u_0^2(x)\,dx\right) + E_0\right)^{\frac{1}{2}} \\
&\leqslant C\big(e^{C_2 T}\big(1 + \|u_0\|\big) + E_0\big)^{\frac{1}{2}} := K_1\big(C, C_2, \|u_0\|, E_0, T\big),
\end{aligned} \tag{3.19}$$

which together with (3.10) and (3.18) implies that

$$\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \leqslant \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + K\big)e^{C_3(t+1)\exp\{C\int_0^t \|\partial_x u(\tau)\|_{L^\infty}\,d\tau\}}, \tag{3.20}$$

where $C_3 = C_3(K_1, \|\rho_0\|_{L^\infty})$.

Hence, if the maximal existence time $T < \infty$ satisfies $\int_0^T \|\partial_x u(\tau)\|_{L^\infty}\, d\tau < \infty$, we obtain from (3.20) that

$$\limsup_{t\to T}\left(\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}\right) < \infty \tag{3.21}$$

contradicts the assumption on the maximal existence time $T < \infty$. This completes the proof of Theorem 3.1 for $s \in (2, 3)$.

Step 2. For $s \in [2, \frac{5}{2})$, applying Proposition 3.2 to the transport equation (3.11), we have

$$\left\|\rho(t)\right\|_{H^{s-1}} \leqslant \|\rho_0\|_{H^{s-1}} + C\int_0^t \|\rho\partial_x u\|_{H^{s-1}}\, d\tau + C\int_0^t \|\rho\|_{H^{s-1}}\|\partial_x u\|_{L^\infty\cap H^{\frac{1}{2}}}\, d\tau.$$

(3.13) (where $s-2$ is replaced by $s-1$) applied implies that

$$\left\|\rho(t)\right\|_{H^{s-1}} \leqslant \|\rho_0\|_{H^{s-1}} + C\int_0^t \|\partial_x u\|_{H^{s-1}}\|\rho\|_{L^\infty}\, d\tau + C\int_0^t \|\rho\|_{H^{s-1}}\|\partial_x u\|_{L^\infty\cap H^{\frac{1}{2}}}\, d\tau,$$

which together with (3.16) yields

$$\begin{aligned}&\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}\\ &\quad\leqslant \|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + K + C\int_0^t \left(\left\|u(\tau)\right\|_{H^s} + \left\|\rho(\tau)\right\|_{H^{s-1}}\right)\left(\|u\|_{H^{\frac{3}{2}+\epsilon_0}} + \left\|\rho(\tau)\right\|_{L^\infty} + 1\right) d\tau,\end{aligned}$$

with $0 < \epsilon_0 < \frac{1}{2}$, where we used the fact $H^{\frac{1}{2}+\epsilon_0} \hookrightarrow L^\infty \cap H^{\frac{1}{2}}$. Applying Gronwall's inequality gives

$$\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}} \leqslant \left(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + K\right)e^{C\int_0^t (\|u\|_{H^{\frac{3}{2}+\epsilon_0}} + \|\rho(\tau)\|_{L^\infty} + 1)\, d\tau}. \tag{3.22}$$

Therefore, using the uniqueness of the solution in Theorem 2.1, (2.11) and (3.21), we get that: if the maximal existence time $T < \infty$ satisfies $\int_0^T \|\partial_x u(\tau)\|_{L^\infty}\, d\tau < \infty$, then (3.22) implies that

$$\limsup_{t\to T}\left(\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}\right) < \infty \tag{3.23}$$

which contradicts the assumption on the maximal existence time $T < \infty$. This completes the proof of Theorem 3.1 for $s \in [2, \frac{5}{2})$.

Step 3. For $s = k \in N$, $k \geqslant 3$, by differentiating (3.11) $k-2$ times with respect to x, we have

$$\partial_t\partial_x^{k-2}\rho + u\partial_x\left(\partial_x^{k-2}\rho\right) + \sum_{l_1+l_2=k-3,\, l_1,l_2\geqslant 0} C_{l_1,l_2}\partial_x^{l_1+1}u\,\partial_x^{l_2+1}\rho + \rho\partial_x\left(\partial_x^{k-2}u\right) = 0. \tag{3.24}$$

Applying Proposition 3.2 to the transport equation (3.24), we have

$$\begin{aligned}\left\|\partial_x^{k-2}\rho(t)\right\|_{H^1} \leqslant{}& \left\|\partial_x^{k-2}\rho_0\right\|_{H^1} + C\int_0^t \left\|\partial_x^{k-2}\rho(\tau)\right\|_{H^1}\left\|\partial_x u(\tau)\right\|_{L^\infty\cap H^{\frac{1}{2}}}\,d\tau\\ &+ C\int_0^t \left\|\Bigg(\sum_{l_1+l_2=k-3,\,l_1,l_2\geqslant 0} C_{l_1,l_2}\partial_x^{l_1+1}u\,\partial_x^{l_2+1}\rho + \rho\,\partial_x\big(\partial_x^{k-2}u\big)\Bigg)(\tau)\right\|_{H^1} d\tau.\end{aligned}$$

Since H^1 is an algebra, we have

$$\left\|\rho\,\partial_x\big(\partial_x^{k-2}u\big)\right\|_{H^1} \leqslant C\|\rho\|_{H^1}\left\|\partial_x^{k-1}u\right\|_{H^1} \leqslant C\|\rho\|_{H^1}\|u\|_{H^s}$$

and

$$\begin{aligned}&\left\|\sum_{l_1+l_2=k-3,\,l_1,l_2\geqslant 0} C_{l_1,l_2}\partial_x^{l_1+1}u\,\partial_x^{l_2+1}\rho\right\|_{H^1}\\ &\quad\leqslant C\sum_{l_1+l_2=k-3,\,l_1,l_2\geqslant 0} C_{l_1,l_2}\left\|\partial_x^{l_1+1}u\right\|_{H^1}\left\|\partial_x^{l_2+1}\rho\right\|_{H^1} \leqslant C\|u\|_{H^{s-1}}\|\rho\|_{H^{s-1}}.\end{aligned}$$

Therefore,

$$\left\|\partial_x^{k-2}\rho(t)\right\|_{H^1} \leqslant \left\|\partial_x^{k-2}\rho_0\right\|_{H^1} + C\int_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}}\big)\big(\|u\|_{H^{s-1}} + \|\rho\|_{H^1}\big)\,d\tau. \tag{3.25}$$

(3.25), together with (3.16) and (3.13) (where $s-2$ is replaced by 1), implies that

$$\begin{aligned}&\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}\\ &\quad\leqslant \|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + K + C\int_0^t \big(\left\|u(\tau)\right\|_{H^s} + \left\|\rho(\tau)\right\|_{H^{s-1}}\big)\big(\left\|u(\tau)\right\|_{H^{s-1}} + \left\|\rho(\tau)\right\|_{H^1} + 1\big)\,d\tau.\end{aligned}$$

Applying Gronwall's inequality yields

$$\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}} \leqslant \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + K\big)e^{C\int_0^t(\|u\|_{H^{s-1}}+\|\rho\|_{H^1}+1)\,d\tau}. \tag{3.26}$$

Therefore, if the maximal existence time $T<\infty$ satisfies $\int_0^T \|\partial_x u(\tau)\|_{L^\infty}\,d\tau<\infty$, using the uniqueness of the solution in Theorem 2.1, we get that

$$\left\|u(t)\right\|_{H^{s-1}} + \left\|\rho(t)\right\|_{H^1}$$

is uniformly bounded by the induction assumption, which together with (3.26) implies

$$\limsup_{t\to T}\big(\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}\big) < \infty.$$

This leads to a contradiction.

Step 4. For $k<s<k+1$ with $k\in N$, $k\geqslant 3$, by differentiating (3.11) $k-1$ times with respect to x, we have

$$\partial_t\partial_x^{k-1}\rho+u\partial_x\big(\partial_x^{k-1}\rho\big)+\sum_{l_1+l_2=k-2,\,l_1,l_2\geqslant 0}C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho+\rho\partial_x\big(\partial_x^{k-1}u\big)=0. \tag{3.27}$$

Proposition 3.3 applied again implies that

$$\begin{aligned}\big\|\partial_x^{k-1}\rho(t)\big\|_{H^{s-k}}\leqslant{}&\big\|\partial_x^{k-1}\rho_0\big\|_{H^{s-k}}+C\int_0^t\big\|\partial_x^{k-1}\rho(\tau)\big\|_{H^{s-k}}\big(\big\|u(\tau)\big\|_{L^\infty}+\big\|\partial_x u(\tau)\big\|_{L^\infty}\big)\,d\tau\\&+C\int_0^t\bigg\|\bigg(\sum_{l_1+l_2=k-2,\,l_1,l_2\geqslant 0}C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho+\rho\partial_x\big(\partial_x^{k-1}u\big)\bigg)(\tau)\bigg\|_{H^{s-k}}\,d\tau.\end{aligned}$$

Using (3.2) and the Sobolev embedding inequality, we have $\forall\epsilon_0\in(0,\frac{1}{2})$

$$\begin{aligned}\big\|\rho\partial_x\big(\partial_x^{k-1}u\big)\big\|_{H^{s-k}}&\leqslant C\big(\|\rho\|_{L^\infty}\big\|\partial_x^k u\big\|_{H^{s-k}}+\|\rho\|_{H^{s-k+1}}\big\|\partial_x^{k-1}u\big\|_{L^\infty}\big)\\&\leqslant C\big(\|\rho\|_{L^\infty}\|u\|_{H^s}+\|\rho\|_{H^{s-k+1}}\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}}\big)\end{aligned}$$

and

$$\begin{aligned}&\bigg\|\sum_{l_1+l_2=k-2,\,l_1,l_2\geqslant 0}C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho\bigg\|_{H^{s-k}}\\&\quad\leqslant C\sum_{l_1+l_2=k-2,\,l_1,l_2\geqslant 0}C_{l_1,l_2}\big(\big\|\partial_x^{l_1+1}u\big\|_{L^\infty}\big\|\partial_x^{l_2+1}\rho\big\|_{H^{s-k}}+\big\|\partial_x^{l_2}\rho\big\|_{L^\infty}\big\|\partial_x^{l_1+1}u\big\|_{H^{s-k+1}}\big)\\&\quad\leqslant C\big(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}}\|\rho\|_{H^{s-k+1}}+\|\rho\|_{H^{k-\frac{3}{2}+\epsilon_0}}\|u\|_{H^s}\big).\end{aligned}$$

Hence,

$$\begin{aligned}&\big\|\partial_x^{k-1}\rho(t)\big\|_{H^{s-k}}\\&\quad\leqslant\big\|\partial_x^{k-1}\rho_0\big\|_{H^{s-k}}+C\int_0^t\big(\big\|u(\tau)\big\|_{H^s}+\big\|\rho(\tau)\big\|_{H^{s-1}}\big)\big(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}}+\|\rho\|_{H^{k-\frac{3}{2}+\epsilon_0}}+1\big)\,d\tau.\end{aligned} \tag{3.28}$$

(3.28), together with (3.16) and (3.13) (where $s-2$ is replaced by $s-k$), implies that

$$\begin{aligned}&\big\|u(t)\big\|_{H^s}+\big\|\rho(t)\big\|_{H^{s-1}}\\&\quad\leqslant\|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}+K+C\int_0^t\big(\big\|u(\tau)\big\|_{H^s}+\big\|\rho(\tau)\big\|_{H^{s-1}}\big)\big(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}}+\|\rho\|_{H^{k-\frac{3}{2}+\epsilon_0}}+1\big)\,d\tau.\end{aligned}$$

Applying Gronwall's inequality yields

$$\begin{aligned}&\left\|u(t)\right\|_{H^s}+\left\|\rho(t)\right\|_{H^{s-1}}\\&\quad\leqslant\left(\|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}+K\right)e^{C\int_0^t(\|u\|_{H^{k-\frac{1}{2}+\epsilon_0}}+\|\rho\|_{H^{k-\frac{3}{2}+\epsilon_0}}+1)\,d\tau}.\end{aligned}\tag{3.29}$$

In consequence, if the maximal existence time $T<\infty$ satisfies $\int_0^T\|\partial_x u(\tau)\|_{L^\infty}\,d\tau<\infty$, using the uniqueness of the solution in Theorem 2.1, we get that

$$\left\|u(t)\right\|_{H^{k-\frac{1}{2}+\epsilon_0}}+\left\|\rho(t)\right\|_{H^{k-\frac{3}{2}+\epsilon_0}}$$

is uniformly bounded by the induction assumption, which implies

$$\limsup_{t\to T}\left(\left\|u(t)\right\|_{H^s}+\left\|\rho(t)\right\|_{H^{s-1}}\right)<\infty,$$

which leads to a contradiction. Therefore, from Step 1 to Step 4, we complete the proof of Theorem 3.1. □

Our next result describes the necessary and sufficient condition for the blow-up of solutions to (1.1).

Theorem 3.2. *Suppose that $\sigma\in\mathbb{R}\setminus\{0\}$. Let $X_0=\binom{u_0}{\rho_0}\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ with $s\geqslant 2$, and let T be the maximal existence time of the solution $X=\binom{u}{\rho}$ to (1.1) with initial data X_0. Then the solution blows up in finite time if and only if*

$$\liminf_{t\to T^-}\left\{\inf_{x\in\mathbb{S}}\sigma u_x(t,x)\right\}=-\infty.\tag{3.30}$$

The approach we take here is the method of characteristics. Applying the following lemma, we may carry out the estimates along the characteristics $\varphi(t,x)$ which captures $\sup_{x\in\mathbb{S}}u_x(t,x)$ and $\inf_{x\in\mathbb{S}}u_x(t,x)$.

Lemma 3.1. *(See [6].) Let $T>0$ and $v\in C^1([0,T);H^2(\mathbb{R}))$. Then for every $t\in[0,T)$, there exists at least one point $\xi(t)\in\mathbb{R}$ with*

$$m(t):=\inf_{x\in\mathbb{S}}v_x(t,x)=v_x\big(t,\xi(t)\big),$$

and the function $m(t)$ is almost everywhere differentiable on $(0,T)$ with

$$\frac{dm}{dt}(t)=v_{tx}\big(t,\xi(t)\big),\quad \text{a.e. on } (0,T).$$

Lemma 3.2. *Let $X_0=\binom{u_0}{\rho_0}\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ with $s\geqslant 2$, and let T be the maximal existence time of the solution $X=\binom{u}{\rho}$ to system (1.1) with initial data X_0. Then:*

(1) $\sigma\neq 0$:

$$\sup_{x\in\mathbb{S}}u_x(t,x)\leqslant\|u_{0,x}\|_{L^\infty(\mathbb{S})}+\sqrt{\frac{\|\rho_0\|^2_{L^\infty(\mathbb{S})}+k_1^2(T)}{\sigma}}\quad(\sigma>0),\tag{3.31}$$

$$\inf_{x\in\mathbb{S}}u_x(t,x)\geqslant-\|u_{0,x}\|_{L^\infty(\mathbb{S})}-\frac{k_2(T)}{\sqrt{-\sigma}}\quad(\sigma<0).\tag{3.32}$$

(2) $\sigma = 0$:

$$\sup_{x\in\mathbb{S}} u_x(t,x) \leqslant \sup_{x\in\mathbb{S}} u_{0,x}(x) + \frac{1}{2}\Big(\sup_{x\in\mathbb{S}} \rho_0^2(x) + k_1^2(T)\Big)t, \tag{3.33}$$

$$\inf_{x\in\mathbb{S}} u_x(t,x) \geqslant \inf_{x\in\mathbb{S}} u_{0,x}(x) + \frac{1}{2}\Big(\sup_{x\in\mathbb{S}} \rho_0^2(x) - k_2^2(T)\Big)t. \tag{3.34}$$

The constants above are defined as follows:

$$k_1(T) = \sqrt{2A + \frac{A}{2}E_0 + \frac{3A}{2}\big[e^{C_2T}\big(\|u_0\|^2_{L^2(\mathbb{S})} + 1\big)\big]}, \tag{3.35}$$

$$k_2(T) = \sqrt{2A + \frac{A+2}{2}E_0 + \frac{3A}{2}\big[e^{C_2T}\big(\|u_0\|^2_{L^2(\mathbb{S})} + 1\big)\big]}. \tag{3.36}$$

Proof of Lemma 3.2. By Theorem 2.1 and a simple density argument, we show the desired results are valid when $s \geqslant 3$, so we take $s = 3$ in the proof.

(1) Let $\sigma > 0$. Using Lemma 3.1 and the fact that

$$\sup_{x\in\mathbb{S}}\big[v_x(t,x)\big] = -\inf_{x\in\mathbb{S}}\big[-v_x(t,x)\big],$$

we can consider $M(t)$ and $\gamma(t)$ as follows,

$$M(t) := u_x\big(t,\xi(t)\big) = \sup_{x\in\mathbb{S}}\big[u_x(t,x)\big], \quad t \in [0,T). \tag{3.37}$$

Hence,

$$u_{xx}\big(t,\xi(t)\big) = 0, \quad \text{a.e. } t \in [0,T). \tag{3.38}$$

Take the trajectory $\varphi(t,x)$ defined in (2.15). Then we know that $\varphi(t,\cdot):\mathbb{R}\to\mathbb{R}$ is a diffeomorphism for every $t \in [0,T)$. Therefore, there exists $x_0(t) \in \mathbb{R}$ such that

$$\varphi\big(t,x_0(t)\big) = \xi(t), \quad t \in [0,T). \tag{3.39}$$

Now let

$$\gamma(t) = \rho\big(t,\varphi(t,x_0)\big), \quad t \in [0,T). \tag{3.40}$$

Therefore, along the trajectory $\varphi(t,x_0)$, Eq. (2.9) and the second equation of (1.1) become

$$\begin{cases} M'(t) = -\dfrac{\sigma}{2}M^2(t) + \dfrac{1}{2}\gamma^2(t) + f\big(t,\varphi(t,x_0)\big), \\ \gamma'(t) = -\gamma M, \quad \text{a.e. } t \in [0,T), \end{cases} \tag{3.41}$$

where the notation $'$ denotes the derivative with respect to t and f represents the function

$$f = -Au + g(t) = -Au - \int_{\mathbb{S}} \left(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au\right)dx. \tag{3.42}$$

We first compute the upper and lower bounds for f for later use in getting the blow-up result,

$$f = -Au - \frac{\sigma}{2}\int_{\mathbb{S}} u_x^2\,dx - \frac{1}{2}\int_{\mathbb{S}} \rho^2\,dx + A\int_{\mathbb{S}} u\,dx \leqslant \frac{A}{2}\left(1+u^2\right) + \frac{A}{2}\left(1+\int_{\mathbb{S}} u^2\,dx\right). \tag{3.43}$$

Since $u^2 \leqslant \frac{1}{2}\int_{\mathbb{S}}(u^2+u_x^2)\,dx$, (2.23), and (2.25), we obtain the upper bound for f

$$\begin{aligned}
f &\leqslant \frac{A}{2}\left(1+\frac{1}{2}\int_{\mathbb{S}}\left(u^2+u_x^2\right)dx\right) + \frac{A}{2}\left(1+\int_{\mathbb{S}} u^2\,dx\right)\\
&\leqslant A + \frac{A}{4}\int_{\mathbb{S}}\left(u_x^2+\rho^2\right)dx + \frac{3A}{4}\int_{\mathbb{S}} u^2\,dx\\
&\leqslant A + \frac{A}{4}E_0 + \frac{3A}{4}\left[e^{C_2 t}\left(\int_{\mathbb{S}} u_0^2(x)\,dx + 1\right)\right]\\
&\leqslant A + \frac{A}{4}E_0 + \frac{3A}{4}\left[e^{C_2 T}\left(\|u_0\|^2_{L^2(\mathbb{S})}+1\right)\right] := \frac{1}{2}k_1^2(T).
\end{aligned} \tag{3.44}$$

Now we turn to the lower bound of f. Using previous arguments, we get

$$\begin{aligned}
-f &= Au + \frac{\sigma}{2}\int_{\mathbb{S}} u_x^2\,dx + \frac{1}{2}\int_{\mathbb{S}} \rho^2\,dx - A\int_{\mathbb{S}} u\,dx\\
&\leqslant \frac{A}{2}\left(1+u^2\right) + \frac{\max(|\sigma|,1)}{2}\int_{\mathbb{S}} u_x^2+\rho^2\,dx + \frac{A}{2}\left(1+\int_{\mathbb{S}} u^2\,dx\right)\\
&\leqslant A + \frac{A+2\max(|\sigma|,1)}{4}\int_{\mathbb{S}} u_x^2+\rho^2\,dx + \frac{3A}{4}\int_{\mathbb{S}} u^2\,dx\\
&\leqslant A + \frac{A+2\max(|\sigma|,1)}{4}E_0 + \frac{3A}{4}\left[e^{C_2 t}\left(\|u_0\|^2_{L^2(\mathbb{S})}+1\right)\right].
\end{aligned} \tag{3.45}$$

When $\sigma < 0$, we have a finer estimate

$$\begin{aligned}
-f &\leqslant A + \frac{A+2}{4}\int_{\mathbb{S}} u_x^2+\rho^2\,dx + \frac{3A}{4}\int_{\mathbb{S}} u^2\,dx\\
&\leqslant A + \frac{A+2}{4}E_0 + \frac{3A}{4}\left[e^{C_2 T}\left(\|u_0\|^2_{L^2(\mathbb{S})}+1\right)\right] := \frac{1}{2}k_2^2(T).
\end{aligned} \tag{3.46}$$

Combining (3.44) and (3.45), we obtain

$$|f| \leqslant A + \frac{A+2\max(|\sigma|,1)}{4}E_0 + \frac{3A}{4}\left[e^{C_2 t}\left(\|u_0\|^2_{L^2(\mathbb{S})}+1\right)\right] := \frac{1}{2}k_3^2(T). \tag{3.47}$$

Since $s \geqslant 3$, we have $u \in C_0^1(\mathbb{S})$. Therefore,

$$\sup_{x\in\mathbb{S}} u_x(t,x) \geqslant 0, \qquad \inf_{x\in\mathbb{S}} u_x(t,x) \leqslant 0, \quad t \in [0,T). \tag{3.48}$$

Hence, $M(t) > 0$ for $t \in [0, T)$. From the second equation of (3.19), we obtain

$$\gamma(t) = \gamma(0) e^{-\int_0^t M(\tau)\,d\tau}. \tag{3.49}$$

Hence,

$$\left|\rho\big(t, \varphi(t, x_0)\big)\right| = \left|\gamma(t)\right| \leqslant \left|\gamma(0)\right| \leqslant \|\rho_0\|_{L^\infty(\mathbb{S})}. \tag{3.50}$$

For any given $x \in \mathbb{S}$, define

$$P_1(t) = M(t) - \|u_{0,x}\|_{L^\infty(\mathbb{S})} - \sqrt{\frac{\|\rho_0\|^2_{L^\infty(\mathbb{S})} + k_1^2(T)}{\sigma}} \quad (\sigma > 0).$$

Observing that $P_1(t)$ is a C^1-differentiable function on $[0, T)$ and satisfies

$$P_1(0) = M(0) - \|u_{0,x}\|_{L^\infty(\mathbb{S})} - \sqrt{\frac{\|\rho_0\|^2_{L^\infty(\mathbb{S})} + k_1^2(T)}{\sigma}} \leqslant M(0) - \|u_{0,x}\|_{L^\infty(\mathbb{S})} \leqslant 0,$$

we now claim

$$P_1(t) \leqslant 0, \quad \forall t \in [0, T).$$

Assume the contrary that there is $t_0 \in [0, T)$ such that $P_1(t_0) > 0$. Let

$$t_1 = \max\big\{t < t_0\colon\ P_1(t) = 0\big\}.$$

Then $P_1(t_1) = 0$ and $P_1'(t_1) \geqslant 0$, or equivalently,

$$M(t_1) = \|u_{0,x}\|_{L^\infty(\mathbb{S})} + \sqrt{\frac{\|\rho_0\|^2_{L^\infty(\mathbb{S})} + k_1^2(T)}{\sigma}}$$

and $M'(t_1) \geqslant 0$ a.e. $t \in [0, T)$. On the other hand, we have

$$\begin{aligned} M'(t_1) &= -\frac{\sigma}{2} M^2(t_1) + \frac{1}{2}\gamma^2(t_1) + f\big(t_1, \varphi(t_1, x)\big) \quad \text{a.e. } t \in [0, T) \\ &\leqslant -\frac{\sigma}{2}\left[\|u_{0,x}\|_{L^\infty(\mathbb{S})} + \sqrt{\frac{\|\rho_0\|^2_{L^\infty(\mathbb{S})} + k_1^2(T)}{\sigma}}\right]^2 + \frac{1}{2}\|\rho_0\|_{L^\infty(\mathbb{S})} + \frac{1}{2}k_1^2(T) < 0, \end{aligned}$$

which is a contradiction. Therefore, $P_1(t) \leqslant 0$ for all $t \in [0, T)$. Since x is arbitrarily chosen, we obtain (3.31).

To derive (3.32) in the case of $\sigma < 0$, we consider $\widetilde{M}(t)$ and $\widetilde{\gamma}(t)$ as in Lemma 3.1,

$$\widetilde{M}(t) := u_x\big(t, \zeta(t)\big) = \inf_{x\in\mathbb{S}}\big[u_x(t, x)\big], \quad t \in [0, T). \tag{3.51}$$

Hence,

$$u_{xx}\big(t, \zeta(t)\big) = 0 \quad \text{a.e. } t \in [0, T). \tag{3.52}$$

Using previous arguments, we take the characteristic $\varphi(t,x)$ defined in (2.15) and choose $x_1(t)\in\mathbb{R}$ such that

$$\varphi\big(t,x_1(t)\big)=\zeta(t). \tag{3.53}$$

Let

$$\widetilde{\gamma}(t)=\rho\big(t,\varphi(t,x_1)\big),\quad t\in[0,T). \tag{3.54}$$

Hence, along the trajectory $\varphi(t,x_1)$, Eq. (2.9) and the second equation of (1.1) become

$$\begin{cases}\widetilde{M}'(t)=-\dfrac{\sigma}{2}\widetilde{M}^2(t)+\dfrac{1}{2}\widetilde{\gamma}^2(t)+f\big(t,\varphi(t,x_1)\big),\\ \widetilde{\gamma}'(t)=-\widetilde{\gamma}\widetilde{M},\quad \text{a.e. } t\in[0,T).\end{cases} \tag{3.55}$$

Define

$$P_2(t)=\widetilde{M}(t)+\|u_{0,x}\|_{L^\infty(\mathbb{S})}+\frac{k_2(T)}{\sqrt{-\sigma}}\quad(\sigma<0),$$

for any given $x\in\mathbb{S}$. Note that $P_2(t)$ is also C^1-differentiable on $[0,T)$ and satisfies

$$P_2(0)=\widetilde{M}(0)+\|u_{0,x}\|_{L^\infty(\mathbb{S})}+\frac{k_2(T)}{\sqrt{-\sigma}}\geqslant\widetilde{M}(0)+\|u_{0x}\|_{L^\infty(\mathbb{S})}\geqslant 0.$$

We now claim that

$$P_2(t)\geqslant 0,\quad \forall t\in[0,T).$$

Suppose not. Then there is $\tilde{t}_0\in[0,T)$ such that $P_2(\tilde{t}_0)<0$. Define

$$t_2=\max\big\{t<\tilde{t}_0\colon\ P_2(t)=0\big\}.$$

Then $P_2(t_2)=0$ and $P_2'(t_2)\leqslant 0$, or equivalently,

$$\widetilde{M}(t_2)=-\|u_{0,x}\|_{L^\infty(\mathbb{S})}-\frac{k_2(T)}{\sqrt{-\sigma}}$$

and $\widetilde{M}'(t_2)\leqslant 0$ a.e. $t\in[0,T)$. However, we have

$$\begin{aligned}\widetilde{M}'(t_2)&=-\frac{\sigma}{2}\widetilde{M}^2(t_2)+\frac{1}{2}\widetilde{\gamma}^2(t_2)+f\big(t_2,\varphi(t_2,x)\big)\quad\text{a.e. } t\in[0,T)\\&\geqslant-\frac{\sigma}{2}\bigg[-\|u_{0,x}\|_{L^\infty(\mathbb{S})}-\frac{k_2(T)}{\sqrt{-\sigma}}\bigg]^2-\frac{1}{2}k_2^2(T)>0,\end{aligned}$$

a contradiction. Therefore, $P_2(t)\geqslant 0$ for $t\in[0,T)$. Since x is chosen arbitrarily, we obtain (3.32).

(2) Let $\sigma=0$. Using previous arguments, Eq. (3.41) becomes

$$\begin{cases}M'(t)=\dfrac{1}{2}\gamma^2(t)+f\big(t,\varphi(t,x_0)\big),\\ \gamma'(t)=-\gamma M,\quad \text{a.e. } t\in[0,T),\end{cases} \tag{3.56}$$

where the notation $'$ denotes the derivative with respect to t and f represents the function

$$f = -Au - \int_{\mathbb{S}} \left(\frac{1}{2}\rho^2 - Au \right) dx. \tag{3.57}$$

We first compute the upper and lower bounds for f for later use in getting the blow-up result,

$$\begin{aligned} f &= -Au - \frac{1}{2}\int_{\mathbb{S}} \rho^2\,dx + A\int_{\mathbb{S}} u\,dx \leqslant \frac{A}{2}\left(1+u^2\right) + \frac{A}{2}\left(1 + \int_{\mathbb{S}} u^2\,dx\right) \\ &\leqslant A + \frac{A}{4}\int_{\mathbb{S}} \left(u_x^2+\rho^2\right) dx + \frac{3A}{4}\int_{\mathbb{S}} u^2\,dx \leqslant A + \frac{A}{4}E_0 + \frac{3A}{4}\left[e^{C_2 t}\left(\int_{\mathbb{S}} u_0^2(x)\,dx + 1 \right) \right] \\ &\leqslant A + \frac{A}{4}E_0 + \frac{3A}{4}\left[e^{C_2 T}\left(\|u_0\|^2_{L^2(\mathbb{S})} + 1 \right) \right]. \end{aligned} \tag{3.58}$$

Now we turn to the lower bound of f:

$$\begin{aligned} -f &\leqslant \frac{A}{2}\left(1+u^2\right) + \frac{1}{2}\int_{\mathbb{S}} \rho^2\,dx + \frac{A}{2}\left(1 + \int_{\mathbb{S}} u^2\,dx\right) \\ &\leqslant A + \frac{A+2}{4}\int_{\mathbb{S}} \left(u_x^2+\rho^2\right) dx + \frac{3A}{4}\int_{\mathbb{S}} u^2\,dx \\ &\leqslant A + \frac{A+2}{4}E_0 + \frac{3A}{4}\left[e^{C_2 t}\left(\int_{\mathbb{S}} u_0^2(x)\,dx + 1 \right) \right] \\ &\leqslant A + \frac{A+2}{4}E_0 + \frac{3A}{4}\left[e^{C_2 T}\left(\|u_0\|^2_{L^2(\mathbb{S})} + 1 \right) \right]. \end{aligned} \tag{3.59}$$

Combining (3.58) and (3.59) we obtain

$$|f| \leqslant A + \frac{A+2}{4}E_0 + \frac{3A}{4}\left[e^{C_2 T}\left(\|u_0\|^2_{L^2(\mathbb{S})} + 1 \right) \right]. \tag{3.60}$$

Since we know $M(t) > 0$ for $t \in [0, T)$, from the second equation of (3.56) we obtain that

$$\gamma(t) = \gamma(0) e^{-\int_0^t M(\tau)\,d\tau}. \tag{3.61}$$

Hence,

$$\left|\rho\left(t, \varphi(t, x_0)\right)\right| = \left|\gamma(t)\right| \leqslant \left|\gamma(0)\right|. \tag{3.62}$$

Therefore, we have

$$\begin{aligned} M'(t) &= \frac{1}{2}\gamma^2(t) + f\left(t, \varphi(t, x_0)\right) \leqslant \frac{1}{2}\gamma^2(0) + \frac{1}{2}k_1^2(T) \\ &\leqslant \left(\sup_{x\in\mathbb{S}} \rho_0^2(x) + k_1^2(T) \right), \quad \text{a.e. } t \in [0, T). \end{aligned} \tag{3.63}$$

Integrating (3.63) on $[0, t]$, we prove (3.33).

To obtain a lower bound for $\inf_{x\in\mathbb{S}} u_x(t,x)$, we use the same argument. Since $\sigma=0$, Eq. (3.56) becomes

$$\begin{cases} \widetilde{M}'(t) = \dfrac{1}{2}\widetilde{\gamma}^2(t) + f\big(t,\varphi(t,x_1)\big), \\ \widetilde{\gamma}'(t) = -\widetilde{\gamma}\widetilde{M}, \quad \text{a.e. } t\in[0,T). \end{cases} \tag{3.64}$$

Because of $\widetilde{M}(t)<0$, we have from the second equation of (3.64) that

$$\widetilde{\gamma}(t) = \widetilde{\gamma}(0)e^{-\int_0^t \widetilde{M}(\tau)\,d\tau}. \tag{3.65}$$

This means that

$$\big|\rho\big(t,\varphi(t,x_1)\big)\big| = \big|\gamma(t)\big| \geqslant \big|\gamma(0)\big|.$$

Then

$$\begin{aligned} \widetilde{M}'(t) &= \frac{1}{2}\widetilde{\gamma}^2(t) + f\big(t,\varphi(t,x_1)\big) \geqslant \frac{1}{2}\widetilde{\gamma}^2(0) + \frac{1}{2}k_2^2(T) \\ &\geqslant \Big(\inf_{x\in\mathbb{S}} \rho_0^2(x) - k_2^2(T)\Big), \quad \text{a.e. } t\in[0,T). \end{aligned} \tag{3.66}$$

Integrating (3.66) on $[0,t]$, we obtain (3.34). This completes the proof of Lemma 3.2. □

Lemma 3.3. *Suppose that $\sigma\in\mathbb{R}\setminus\{0\}$. Let $X_0=\binom{u_0}{\rho_0}\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ with $s\geqslant 2$, and let T be the maximal existence time of the solution $X=\binom{u}{\rho}$ to (1.1) with initial data X_0. Then we have*

$$\rho\big(t,\varphi(t,x)\big)\varphi_x(t,x) = \rho_0(x), \quad \forall (t,x)\in[0,T)\times\mathbb{S}. \tag{3.67}$$

Moreover, if there exists $M>0$ such that

$$\inf_{(t,x)\in[0,T)\times\mathbb{S}} \sigma u_x(t,x) \geqslant -M, \quad \forall (t,x)\in[0,T)\times\mathbb{S}, \tag{3.68}$$

then

$$\big\|\rho(t,\cdot)\big\|_{L^\infty(\mathbb{S})} = \big\|\rho\big(t,\varphi(t,\cdot)\big)\big\|_{L^\infty(\mathbb{S})} \leqslant e^{MT/\sigma}\big\|\rho_0(\cdot)\big\|_{L^\infty(\mathbb{S})} \quad (\sigma>0), \tag{3.69}$$

$$\big\|\rho(t,\cdot)\big\|_{L^\infty(\mathbb{S})} = \big\|\rho\big(t,\varphi(t,\cdot)\big)\big\|_{L^\infty(\mathbb{S})} \leqslant e^{NT}\big\|\rho_0(\cdot)\big\|_{L^\infty(\mathbb{S})} \quad (\sigma<0) \tag{3.70}$$

where $N=\|u_{0,x}\|_{L^\infty(\mathbb{S})}+\frac{k_2(T)}{\sqrt{-\sigma}}$ and $k_2(T)$ is given in (3.36).

Proof. Differentiating the left-hand side of Eq. (3.67) with respect to t, in view of the relations (2.15) and (1.1), we obtain

$$\begin{aligned} \frac{d}{dt}\big\{\rho\big(t,\varphi(t,x)\big)\varphi_x(t,x)\big\} &= \big[\rho_t(t,\varphi)+\rho_x(t,\varphi)\varphi_t(t,x)\big]\varphi_x(t,x) + \rho(t,\varphi)\varphi_{xt}(t,x) \\ &= \big[\rho_t(t,\varphi)+\rho_x(t,\varphi)u(t,\varphi)\big]\varphi_x(t,x) + \rho(t,\varphi)u_x(t,\varphi)\varphi_x(t,x) \\ &= \big[\rho_t(t,\varphi)+\rho_x(t,\varphi)u(t,\varphi)+\rho(t,\varphi)u_x(t,\varphi)\big]\varphi_x(t,x) = 0. \end{aligned}$$

This completes the proof of (3.67). In view of the assumption (3.68) and $\sigma > 0$, we obtain

$$u(t,x) \geqslant -\frac{M}{\sigma}, \quad \forall (t,x) \in [0,T) \times \mathbb{S}.$$

By Lemma 2.2 and (3.67), we have

$$\left\|\rho(t,\cdot)\right\|_{L^\infty(\mathbb{S})} = \left\|\rho\big(t,\varphi(t,\cdot)\big)\right\|_{L^\infty(\mathbb{S})} = \left\|e^{-\int_0^t u_x(\tau,\cdot)\,d\tau}\rho_0(\cdot)\right\|_{L^\infty(\mathbb{S})} \leqslant e^{MT/\sigma}\left\|\rho_0(\cdot)\right\|_{L^\infty(\mathbb{S})}.$$

To obtain (3.70), we use a similar argument as before. Using (2.16), (3.67), and the lower bound for $u_x(t,x)$ in (3.32), it follows that

$$\left\|\rho(t,\cdot)\right\|_{L^\infty(\mathbb{S})} = \left\|\rho\big(t,\varphi(t,\cdot)\big)\right\|_{L^\infty(\mathbb{S})} = \left\|e^{-\int_0^t u_x(\tau,\cdot)\,d\tau}\rho_0(\cdot)\right\|_{L^\infty(\mathbb{S})} \leqslant e^{NT}\left\|\rho_0(\cdot)\right\|_{L^\infty(\mathbb{S})},$$

which proves (3.70). This completes the proof of Lemma 3.3. □

Proof of Theorem 3.2. Suppose that $T < \infty$ and (3.30) is not valid. Then there is some positive number $M > 0$ such that

$$\sigma u_x(t,x) \geqslant -M, \quad \forall (t,x) \in [0,T) \times \mathbb{S}.$$

It now follows from Lemma 3.2 that $|u_x(t,x)| \leqslant C$, where $C = C(A, M, \sigma, E_0, \|u_0\|, T)$. Therefore, Theorem 3.1 implies that the maximal existence time $T = \infty$, which contradicts the assumption that $T < \infty$.

Conversely, the Sobolev embedding theorem $H^s(\mathbb{S}) \hookrightarrow L^\infty(\mathbb{S})$ with $s > \frac{1}{2}$ implies that if (3.30) holds, the corresponding solution blows up in finite time, which completes the proof of Theorem 3.2. □

4. Wave-breaking data and blow-up rate

Now we will give our first wave-breaking result.

Theorem 4.1. *Let $\sigma \in \mathbb{R} \setminus \{0\}$. Suppose $X_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geqslant 2$, and let T be the maximal existence time of the corresponding solution to (1.1) with the initial data X_0.*

(i) $\sigma > 0$: *If there is some $\bar{x} \in \mathbb{S}$ such that $\rho_0(\bar{x}) = 0$, $u_{0,x}(\bar{x}) = \inf_{x\in\mathbb{S}} u_{0,x}(x)$, and $u_{0,x}(\bar{x}) < -\frac{k_1(T)}{\sqrt{\sigma}}$, then the corresponding solution to (1.1) blows up in finite time T_1 with*

$$0 < T_1 \leqslant -\frac{2}{\sigma u_{0,x}(\bar{x}) + \sqrt{-k_1(T)\sigma^{3/2}u_{0,x}(\bar{x})}}, \tag{4.1}$$

such that $\liminf_{t\to T_1^-}(\inf_{x\in\mathbb{S}} u_x(t,x)) = -\infty$.

(ii) $\sigma < 0$: *If there is some $\bar{x} \in \mathbb{S}$ such that $u_{0,x}(\bar{x}) > \frac{k_2(T)}{\sqrt{-\sigma}}$, then the corresponding solution to (1.1) blows up in finite time T_2 with*

$$0 < T_2 \leqslant -\frac{2}{\sigma u_{0,x}(\bar{x}) + \sqrt{-k_2(T)\sigma^{3/2}u_{0,x}(\bar{x})}}, \tag{4.2}$$

such that $\liminf_{t\to T_2^-}(\sup_{x\in\mathbb{S}} u_x(t,x)) = \infty$.

Proof. (i) Let $\sigma > 0$. We use a similar argument to the proof of Lemma 3.2. So we take $s \geqslant 3$. We consider along the trajectory $\varphi(t, x_1)$ defined in (2.15) and (3.53). In this way, we can write the transport equation of ρ in (1.1) along the trajectory of $\varphi(t, x_1)$ as

$$\frac{d}{dt}\rho\big(t, \zeta(t)\big) = -\rho\big(t, \zeta(t)\big)u_x\big(t, \zeta(t)\big). \tag{4.3}$$

From the assumption of the theorem, we see

$$\widetilde{M}(0) = u_x\big(0, \zeta(0)\big) = \inf_{x\in\mathbb{S}} u_{0,x}(x) = u_{0,x}(\bar{x}).$$

Hence, we can choose $\zeta(0) = \bar{x}$ and then $\rho(\zeta(0)) = \rho(\bar{x}) = 0$. Thus, from (4.3) we see that

$$\rho\big(t, \zeta(t)\big) = 0 \quad \forall t \in [0, T). \tag{4.4}$$

Using the upper bound of f in (3.44) and (4.4), we obtain

$$\widetilde{M}'(t) \leqslant -\frac{\sigma}{2}\widetilde{M}^2(t) + \frac{1}{2}k_1^2(T), \quad \text{a.e. } t \in [0, T). \tag{4.5}$$

If $u_{0,x}(\bar{x}) < -\frac{k_1(T)}{\sqrt{\sigma}}$, then $\widetilde{M}(0) < -\frac{k_1(T)}{\sqrt{\sigma}}$. Hence, $\widetilde{M}'(0) < 0$ and $\widetilde{M}(t)$ is strictly decreasing for all $t \in [0, T)$. Define

$$\omega := \frac{1}{2} - \frac{1}{2}\sqrt{\frac{k_1(T)}{-u_{0,x}(\bar{x})\sqrt{\sigma}}} \in \left(0, \frac{1}{2}\right).$$

Using that $\widetilde{M}(t) < \widetilde{M}(0) = u_{0,x}(\bar{x}) < 0$, we obtain

$$\widetilde{M}'(t) \leqslant -\frac{\sigma}{2}\widetilde{M}^2(t) + \frac{1}{2}k_1^2(T) \leqslant -\frac{\sigma}{2}\widetilde{M}^2(t)\big[1 - (1-2\omega)^4\big] \leqslant -\omega\sigma\widetilde{M}^2(t) \quad \text{a.e. } t \in [0, T).$$

By solving the above inequality, we conclude that

$$\widetilde{M}(t) \leqslant \frac{u_{0,x}(\bar{x})}{1 + \omega\sigma u_{0,x}(\bar{x})t} \to -\infty, \quad \text{as } t \to -\frac{1}{\omega\sigma u_{0,x}(\bar{x})}.$$

Hence,

$$T \leqslant -\frac{1}{\omega\sigma u_{0,x}(\bar{x})},$$

which proves (4.1).

(ii) Similarly as in the case $\sigma > 0$, we consider the functions $M(t)$ and $\xi(t)$ as defined in (3.37), and we take the trajectory $\varphi(t, x_0)$ with x_0 defined in (3.39). Then we have

$$\begin{aligned} M'(t) &= -\frac{\sigma}{2}M^2(t) + \frac{1}{2}\rho^2\big(t, \xi(t)\big) + f\big(t, \varphi(t, x_0)\big) \\ &\geqslant -\frac{\sigma}{2}M^2(t) + f\big(t, \varphi(t, x_0)\big), \quad \text{a.e. } t \in [0, T). \end{aligned} \tag{4.6}$$

Using the lower bound of f as in (3.46), we obtain

$$M'(t) \geqslant -\frac{\sigma}{2}M^2(t) - \frac{1}{2}k_2^2(T), \quad \text{a.e. } t \in [0, T). \tag{4.7}$$

By assumption of the theorem, we have $M(0) \geqslant u_{0,x}(\bar{x}) > \frac{k_2(T)}{\sqrt{-\sigma}}$. This implies that $M'(0) > 0$ and $M(t)$ is strictly increasing for all $t \in [0, T)$. Define

$$\delta := \frac{1}{2} + \frac{1}{2}\sqrt{\frac{k_2(T)}{u_{0,x}(\bar{x})\sqrt{-\sigma}}} \in \left(\frac{1}{2}, 1\right).$$

Using that $M(t) > M(0) = u_{0,x}(\bar{x}) > 0$, we have

$$M'(t) \geqslant -\frac{\sigma}{2}M^2(t) - \frac{1}{2}k_2^2(T) \geqslant -\frac{\sigma}{2}M^2(t)\left[1 - (2\delta - 1)^4\right] \geqslant -\delta\sigma M^2(t) \quad \text{a.e. } t \in [0, T).$$

Therefore,

$$M(t) \geqslant \frac{u_{0,x}(\bar{x})}{1 + \delta\sigma u_{0,x}(\bar{x})t} \to \infty, \quad \text{as } t \to -\frac{1}{\delta\sigma u_{0,x}(\bar{x})}.$$

Hence,

$$T \leqslant -\frac{1}{\delta\sigma u_{0,x}(\bar{x})},$$

which proves (4.2). □

In the following theorem, we are interested in the wave-breaking phenomenon when the initial value is odd and even.

Theorem 4.2. *Let $\sigma \in \mathbb{R} \setminus \{0\}$, and $A = 0$. Suppose $X_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geqslant 2$, and let T be the maximal existence time of the corresponding solution to* (1.1) *with the initial data X_0.*

(i) $\sigma > 0$: *If u_0 is odd with $u_{0,x}(0) < 0$ and ρ_0 is even with $\rho_0(0) = 0$, then the corresponding solution to* (1.1) *blows up in finite time T_1 with*

$$0 < T_1 \leqslant -\frac{2}{\sigma u_{0,x}(0)}$$

such that $\lim_{t \to T_1^-} u_x(t, 0) = -\infty$.

(ii) $\sigma < 0$: *If u_0 is odd, ρ_0 is even and $u_{0,x}(0) > \sqrt{\frac{E_0}{-\sigma}}$, then the corresponding solution to* (1.1) *blows up in finite time T_2 with*

$$0 < T_2 \leqslant -\frac{2}{\sigma u_{0,x}(0) + \sqrt{-\sigma^{3/2} u_{0,x}(0)\sqrt{E_0}}},$$

such that $\lim_{t \to T_2^-} u_x(t, 0) = \infty$.

Proof. First, we notice that if $A=0$ in the first equation of (1.1) then $u(t,x)$ is odd and $\rho(t,x)$ is even, due to the algebraic structure of the first equation in (1.1). Hence $u(t,0)=0$ and $\rho_x(t,0)=0$.

(i) Observe next that $\rho(t,0)=0$ for all times of existence. Indeed, one has

$$\rho_t(t,0)=-u\rho_x(t,0)-u_x\rho(t,0).$$

Note that the first term on the right-hand side vanished since $u(t,0)=0$ and $\rho_x(t,0)=0$. Together with the assumption $\rho_0(0)=0$, it means that $\rho(t,0)=0$. Evaluating (2.9) at $(t,0)$ and denoting $M(t)=u_x(t,0)$, we obtain

$$M'(t)+\frac{\sigma}{2}M^2(t)=g(t), \quad \text{a.e. } t\in[0,T). \tag{4.8}$$

Using $A=0$ and $\sigma>0$ in (2.10), we know that $g(t)\leqslant 0$. Thus, we have

$$M'(t)+\frac{\sigma}{2}M^2(t)\leqslant 0, \quad \text{a.e. } t\in[0,T). \tag{4.9}$$

Thus, if $u_{0,x}(0)<0$ holds, namely, $M(0)<0$, then $M(t)<0$ for all $t\in[0,T)$ and

$$-\frac{1}{M(t)}+\frac{1}{M(0)}\leqslant -\frac{\sigma}{2}t.$$

This implies

$$u_x(t,0)=M(t)\leqslant \frac{2M(0)}{2+\sigma M(0)t}\to-\infty, \quad \text{as } t\to-\frac{2}{\sigma M(0)}. \tag{4.10}$$

(ii) Using previous arguments, evaluating (2.9) at $(t,0)$ and denoting $M(t)=u_x(t,0)$, we obtain

$$M'(t)+\frac{\sigma}{2}M^2(t)=\frac{1}{2}\rho^2(t,0)+g(t)\geqslant g(t), \quad \text{a.e. } t\in[0,T). \tag{4.11}$$

Using $A=0$ and $\sigma<0$ in (2.10), we know that $g(t)\geqslant-\frac{1}{2}E_0$. Thus, from (4.8), we get

$$M'(t)\geqslant-\frac{\sigma}{2}M^2(t)-\frac{1}{2}E_0, \quad \text{a.e. } t\in[0,T). \tag{4.12}$$

By assumption, $u_{0,x}(0)>\sqrt{\frac{E_0}{-\sigma}}$, $M(0)=u_{0,x}(0)>\sqrt{\frac{E_0}{-\sigma}}$. We see that $M(0)>0$ and $M(t)$ is strictly increasing over $[0,T)$. Define

$$\theta:=\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{u_{0,x}(0)}\sqrt{\frac{E_0}{-\sigma}}}\in\left(\frac{1}{2},1\right).$$

Using $M(t)>M(0)=u_{0,x}(0)>\sqrt{\frac{E_0}{-\sigma}}>0$, we have

$$M'(t)\geqslant-\frac{\sigma}{2}M^2(t)-\frac{1}{2}E_0\geqslant-\frac{\sigma}{2}M^2(t)\big[1-(2\theta-1)^4\big]\geqslant-\theta\sigma M^2(t) \quad \text{a.e. } t\in[0,T).$$

Therefore,

$$M(t) \geqslant \frac{u_{0,x}(0)}{1+\theta\sigma u_{0,x}(0)t} \to \infty, \quad \text{as } t \to -\frac{1}{\theta\sigma u_{0,x}(0)}.$$

This completes the proof of Theorem 4.2. □

Our attention is now turned to the question of the blow-up rate of the slope to a breaking wave for (1.1).

Theorem 4.3. *Let $\sigma \in \mathbb{R}\setminus\{0\}$. If $T<\infty$ is the blow-up time of the solution to system* (1.1) *with the initial data X_0 with $s \geqslant 2$ satisfying the assumption of Theorem* 4.1, *then*

$$\lim_{t\to T^-}\Big[\Big(\inf_{x\in\mathbb{S}} u_x(t,x)\Big)(T-t)\Big] = -\frac{2}{\sigma} \quad (\sigma>0), \tag{4.13}$$

$$\lim_{t\to T^-}\Big[\Big(\sup_{x\in\mathbb{S}} u_x(t,x)\Big)(T-t)\Big] = -\frac{2}{\sigma} \quad (\sigma<0). \tag{4.14}$$

Proof. We may assume $s=3$ to prove the theorem. Let $\sigma>0$. Using (3.47), (4.4) and denoting $K(T)=\frac{1}{2}k_3^2(T)$, we know

$$-\frac{\sigma}{2}\widetilde{M}^2(t) - K(T) \leqslant \widetilde{M}'(t) \leqslant -\frac{\sigma}{2}\widetilde{M}^2(t) + K(T), \quad \text{a.e. } t\in[0,T). \tag{4.15}$$

Now fix any $\epsilon \in (0,\sigma/2)$. Since $\widetilde{M}(t) \to -\infty$ as $t\to T^-$, there exists $t_0\in(0,T)$ such that $\widetilde{M}(t_0) < -\sqrt{2\sigma K(T)+\frac{K(T)}{\epsilon}}$. Notice that $\widetilde{M}(t)$ is locally Lipschitz so that $\widetilde{M}(t)$ is absolutely continuous on $[0,T)$. It then follows from (4.15) that $\widetilde{M}(t)$ is decreasing on $[t_0,T)$ and satisfies

$$\widetilde{M}(t) < -\sqrt{2\sigma K(T)+\frac{K(T)}{\epsilon}} < -\sqrt{\frac{K(T)}{\epsilon}}, \quad t\in[t_0,T).$$

Then (4.15) implies that

$$\frac{\sigma}{2}-\epsilon < \frac{d}{dt}\left(\frac{1}{\widetilde{M}(t)}\right) < \frac{\sigma}{2}+\epsilon, \quad \text{a.e. } t\in[t_0,T).$$

Integrating the above equation on (t,T) with $t\in(t_0,T)$ and noticing that $\widetilde{M}(t)\to-\infty$ as $t\to T^-$, we obtain

$$\left(\frac{\sigma}{2}-\epsilon\right)(T-t) \leqslant -\frac{1}{\widetilde{M}(t)} \leqslant \left(\frac{\sigma}{2}+\epsilon\right)(T-t).$$

Since $\epsilon\in(0,\sigma/2)$ is arbitrary, in view of the definition of $\widetilde{M}(t)$, the above inequality implies (4.13).

When $\sigma<0$, from (4.6), we have

$$M'(t) \geqslant -\frac{\sigma}{2}M^2(t) - K(T), \quad \text{a.e. } t\in[0,T).$$

Since $M(t) \to \infty$ as $t \to T^-$, there exists $t_0 \in (0, T)$ such that $M(t_0) < \sqrt{-2\sigma K(T)}$. Therefore, we have that $M(t)$ is strictly increasing on $[t_0, T)$ and $M(t) > M(t_0) > \sqrt{-2\sigma K(T)} > 0$. Using the transport equation for ρ, we have that

$$\rho'\big(t, \xi(t)\big) = -M(t)\rho\big(t, \xi(t)\big).$$

Hence,

$$\rho\big(t, \xi(t)\big) = \rho\big(t_0, \xi(t_0)\big)e^{-\int_{t_0}^{t} M(\tau)\,d\tau}, \quad t \in [t_0, T).$$

Then

$$\rho^2\big(t, \xi(t)\big) \leqslant \rho^2\big(t_0, \xi(t_0)\big), \quad t \in [t_0, T).$$

Therefore, using (4.6) again, we have

$$-\frac{\sigma}{2}M^2(t) - \frac{1}{2}\rho^2\big(t_0, \xi(t_0)\big) - K(T) \leqslant M'(t) \leqslant -\frac{\sigma}{2}M^2(t) + \frac{1}{2}\rho^2\big(t_0, \xi(t_0)\big) + K(T). \quad (4.16)$$

Now let $\widetilde{K}(T) = \frac{1}{2}\rho^2(t_0, \xi(t_0)) + K(T)$, and choose $\epsilon \in (0, -\sigma/2)$. We can pick $t_1 \in [t_0, T)$ such that $M(t_1) > \sqrt{-2\sigma\widetilde{K}(T) + \frac{\widetilde{K}(T)}{\epsilon}}$. Then

$$M(t) > M(t_1) > \sqrt{-2\sigma\widetilde{K}(T) + \frac{\widetilde{K}(T)}{\epsilon}} > \sqrt{\frac{\widetilde{K}(T)}{\epsilon}}.$$

Hence, (4.16) implies that

$$\frac{\sigma}{2} - \epsilon < \frac{d}{dt}\left(\frac{1}{M(t)}\right) < \frac{\sigma}{2} + \epsilon, \quad \text{a.e. } t \in [t_1, T).$$

Integrating the above equation on (t, T) with $t \in (t_1, T)$ and noticing that $M(t) \to \infty$ as $t \to T^-$, we obtain

$$\left(\frac{\sigma}{2} - \epsilon\right)(T - t) \leqslant -\frac{1}{M(t)} \leqslant \left(\frac{\sigma}{2} + \epsilon\right)(T - t).$$

Since $\epsilon \in (0, -\sigma/2)$ is arbitrary, in view of the definition of $M(t)$, the above inequality implies (4.14). $\square$

5. Global existence

In this section, we provide a sufficient condition for the global solution of (1.1) in the case when $0 < \sigma < 2$ and $\sigma = 0$.

Theorem 5.1. *Suppose that* $0 < \sigma < 2$. *Let* $X_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ *with* $s \geqslant 2$ *and let T be the maximal time of existence. If we further assume that*

$$\inf_{x \in \mathbb{S}} \rho_0(x) > 0, \quad (5.1)$$

then the solution $X = \binom{u}{\rho}$ *to* (1.1) *corresponding to* X_0 *is global.*

Proof. Using previous arguments, a density argument indicates that it suffices to prove the desired results for $s \geqslant 3$. Thus, we have

$$\inf_{x\in\mathbb{S}} u_x(t,x) \leqslant 0, \qquad \sup_{x\in\mathbb{S}} u_x(t,x) \geqslant 0, \quad t \in [0,T),$$

as before. It suffices to get some uniform a priori estimates for the solution X.

First we will estimate $|\inf_{x\in\mathbb{S}} u_x(t,x)|$. Define $\widetilde{M}(t)$ and $\zeta(t)$ as in (3.51), and consider along the characteristics $\varphi(t, x_0(t))$ as in (2.15) and (3.39).

Thus, from (3.48),

$$\widetilde{M}(t) \leqslant 0 \quad \forall t \in [0,T). \tag{5.2}$$

Letting $\widetilde{\gamma}(t) = \rho(t,\zeta(t))$ and evaluating (2.9) and the second equation of (1.1) at $(t,\zeta(t))$, we have

$$\begin{cases} \widetilde{M}'(t) = -\dfrac{\sigma}{2}\widetilde{M}^2(t) + \dfrac{1}{2}\widetilde{\gamma}^2(t) + f\big(t,\varphi(t,x_0)\big), \\ \widetilde{\gamma}'(t) = -\widetilde{\gamma}(t)\widetilde{M}(t), \quad \text{a.e. } t \in [0,T), \end{cases} \tag{5.3}$$

where f is defined in (3.42). The second equation implies that $\widetilde{\gamma}(t)$ and $\widetilde{\gamma}(0)$ are of the same sign.

Inspired by [7] (see also [5]), we now construct a Lyapunov function for (1.1). Due to a free parameter σ, we could not find a uniform Lyapunov function. Instead, we will divide the case $0 < \sigma \leqslant 1$ and the case $1 < \sigma < 2$. From (5.1), we know that $\widetilde{\gamma}(0) = \rho(0,\zeta(0)) > 0$.

For $0 < \sigma \leqslant 1$, define the following strictly positive Lyapunov function

$$\widetilde{w}(t) = \widetilde{\gamma}(0)\widetilde{\gamma}(t) + \frac{\widetilde{\gamma}(0)}{\widetilde{\gamma}(t)}\big(1 + \widetilde{M}^2(t)\big). \tag{5.4}$$

Computing the evolution of $\widetilde{w}$ and using (5.3), we get

$$\begin{aligned} \widetilde{w}'(t) &= \widetilde{\gamma}(0)\widetilde{\gamma}'(t) - \frac{\widetilde{\gamma}(0)}{\widetilde{\gamma}(t)}\widetilde{\gamma}'(t)\big[1 + \widetilde{M}^2(t)\big] + 2\frac{\widetilde{\gamma}(0)}{\widetilde{\gamma}(t)}\widetilde{M}(t)\widetilde{M}'(t) \\ &= \frac{2\widetilde{\gamma}(0)\widetilde{M}(t)}{\widetilde{\gamma}(t)}\left[\frac{1-\sigma}{2}\widetilde{M}^2(t) + \frac{1}{2} + f\big(t,\varphi(t,x_0)\big)\right] \\ &\leqslant \frac{\widetilde{\gamma}(0)}{\widetilde{\gamma}(t)}\big(1 + \widetilde{M}^2(t)\big)\left[\big|f\big(t,\varphi(t,x_0)\big)\big| + \frac{1}{2}\right] \\ &\leqslant \left[\frac{1}{2} + \frac{1}{2}k_3^2(T)\right]\widetilde{w}(t), \end{aligned} \tag{5.5}$$

where we have used (5.2) and the bound (3.47) for f.

By Gronwall's inequality, we obtain

$$\widetilde{w}(t) \leqslant \widetilde{w}(0)e^{[\frac{1}{2}+\frac{1}{2}k_3^2(T)]t} \leqslant \big(1 + \|\rho_0\|_{L^\infty}^2 + \|u_{0,x}\|_{L^\infty}^2\big)e^{[\frac{1}{2}+\frac{1}{2}k_3^2(T)]t}. \tag{5.6}$$

Recalling that $\widetilde{\gamma}(t)$ and $\widetilde{\gamma}(0)$ are of the same sign, we have

$$\widetilde{\gamma}(0)\widetilde{\gamma}(t) \leqslant \widetilde{w}(t), \qquad \big|\widetilde{\gamma}(0)\big|\big|\widetilde{M}(t)\big| \leqslant \widetilde{w}(t).$$

Then from (5.6), we have

$$\left|\inf_{x\in\mathbb{S}} u_x(t,x)\right| = \left|\widetilde{M}(t)\right| \leqslant \frac{\widetilde{w}(t)}{\widetilde{\gamma}(0)} \leqslant \frac{(1+\|\rho_0\|_{L^\infty}^2+\|u_{0,x}\|_{L^\infty}^2)}{\inf_{x\in\mathbb{S}}\rho_0(x)} e^{[\frac{1}{2}+\frac{1}{2}k_3^2(T)]t}. \tag{5.7}$$

For $1 \leqslant \sigma < 2$, we may define the strictly positive Lyapunov function to be

$$\widetilde{\nu}(t) = \frac{\widetilde{\gamma}^\sigma(0)}{\widetilde{\gamma}^\sigma(t)}\left[\widetilde{\gamma}^2(t)+\widetilde{M}^2(t)+1\right]. \tag{5.8}$$

Differentiating $\widetilde{\nu}(t)$ and using (5.2), we obtain

$$\begin{aligned}
\widetilde{\nu}'(t) &= \frac{2\widetilde{\gamma}^\sigma(0)\widetilde{M}(t)}{\widetilde{\gamma}^\sigma(t)}\left[\frac{\sigma-1}{2}\widetilde{\gamma}^2(t)+\frac{\sigma}{2}+f\big(t,\varphi(t,x_0)\big)\right] \\
&\leqslant \frac{\widetilde{\gamma}^\sigma(0)}{\widetilde{\gamma}^\sigma(t)}\big(1+\widetilde{M}^2(t)\big)\left[\frac{\sigma}{2}+\left|f\big(t,\varphi(t,x_0)\big)\right|\right] \\
&\leqslant \left[\frac{\sigma}{2}+\frac{1}{2}k_3^2(T)\right]\widetilde{\nu}(t).
\end{aligned} \tag{5.9}$$

Thus,

$$\widetilde{\nu}(t) \leqslant \widetilde{\nu}(0)e^{[\frac{\sigma}{2}+\frac{1}{2}k_3^2(T)]t} \leqslant \big(1+\|\rho_0\|_{L^\infty}^2+\|u_{0,x}\|_{L^\infty}^2\big)e^{[\frac{\sigma}{2}+\frac{1}{2}k_3^2(T)]t}. \tag{5.10}$$

Applying Young's inequality ($ab \leqslant a^p/p + b^q/q$) to (5.8) with

$$p = \frac{2}{\sigma}, \qquad q = \frac{2}{2-\sigma},$$

we have

$$\begin{aligned}
\frac{\widetilde{\nu}(t)}{\widetilde{\gamma}^\sigma(0)} &= \left[\widetilde{\gamma}^{\frac{\sigma(2-\sigma)}{2}}\right]^{2/\sigma} + \left[\frac{(1+\widetilde{M}^2)^{\frac{2-\sigma}{2}}}{\widetilde{\gamma}^{\frac{\sigma(2-\sigma)}{2}}}\right]^{2/(2-\sigma)} \\
&\geqslant \frac{\sigma}{2}\left[\widetilde{\gamma}^{\frac{\sigma(2-\sigma)}{2}}\right]^{2/\sigma} + \frac{2-\sigma}{2}\left[\frac{(1+\widetilde{M}^2)^{\frac{2-\sigma}{2}}}{\widetilde{\gamma}^{\frac{\sigma(2-\sigma)}{2}}}\right]^{2/(2-\sigma)} \\
&\geqslant \big(1+\widetilde{M}^2\big)^{\frac{2-\sigma}{2}} \geqslant \left|\widetilde{M}(t)\right|^{2-\sigma}.
\end{aligned}$$

Therefore,

$$\left|\inf_{x\in\mathbb{S}} u_x(t,x)\right| \leqslant \left[\frac{\widetilde{\nu}(t)}{\widetilde{\gamma}^\sigma(0)}\right]^{\frac{1}{2-\sigma}} \leqslant \frac{(1+\|\rho_0\|_{L^\infty}^2+\|u_{0x}\|_{L^\infty}^2)^{\frac{1}{2-\sigma}}}{\inf_{x\in\mathbb{S}}\rho_0^{\frac{\sigma}{2-\sigma}}(x)} e^{\frac{[\frac{\sigma}{2}+\frac{1}{2}k_3^2(T)]t}{2-\sigma}}. \tag{5.11}$$

Next we try to control $|\sup_{x\in\mathbb{S}} u_x(t,x)|$. Similarly as before, we consider $M(t)$, $\xi(t)$ and $\varphi(t,x_1(t))$ as in (3.37) and (3.53). Then (5.3) becomes

$$\begin{cases}
M'(t) = -\dfrac{\sigma}{2}M^2(t) + \dfrac{1}{2}\gamma^2(t) + f\big(t,\varphi(t,x_1)\big), \\
\gamma'(t) = -\gamma(t)M(t), \quad \text{a.e. } t\in[0,T),
\end{cases} \tag{5.12}$$

where $\gamma(t) = \rho(t,\xi(t))$. It follows from (3.48) that

$$M(t) \geqslant 0 \quad \forall t \in [0, T). \tag{5.13}$$

For $0 < \sigma \leqslant 1$, the corresponding Lyapunov function is

$$w(t) = \frac{\gamma^{\sigma}(0)}{\gamma^{\sigma}(t)}\left[\gamma^2(t) + M^2(t) + 1\right]. \tag{5.14}$$

Then from (5.9) and (5.13), we see that

$$w'(t) \leqslant \left[\frac{\sigma}{2} + \frac{1}{2}k_3^2(T)\right]w(t).$$

This implies $w(t) \leqslant (1 + \|\rho_0\|_{L^\infty}^2 + \|u_{0,x}\|_{L^\infty}^2)e^{[\frac{\sigma}{2}+\frac{1}{2}k_3^2(T)]t}$.

Hence, by using previous arguments, we get

$$\frac{w(t)}{\gamma^{\sigma}(0)} \geqslant \left|M(t)\right|^{2-\sigma}.$$

Therefore,

$$\left|\sup_{x\in\mathbb{S}} u_x(t,x)\right| \leqslant \left[\frac{w(t)}{\gamma^{\sigma}(0)}\right]^{\frac{1}{2-\sigma}} \leqslant \frac{(1 + \|\rho_0\|_{L^\infty}^2 + \|u_{0,x}\|_{L^\infty}^2)^{\frac{1}{2-\sigma}}}{\inf_{x\in\mathbb{S}} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} e^{\frac{[\frac{\sigma}{2}+\frac{1}{2}k_3^2(T)]t}{2-\sigma}}. \tag{5.15}$$

For $1 \leqslant \sigma < 2$, consider the Lyapunov function

$$\nu(t) = \gamma(0)\gamma(t) + \frac{\gamma(0)}{\gamma(t)}\left(1 + M^2(t)\right). \tag{5.16}$$

From (5.13) and (5.5),

$$\nu'(t) \leqslant \left[\frac{1}{2} + \frac{1}{2}k_3^2(T)\right]\nu(t),$$

then $\nu(t) \leqslant (1 + \|\rho_0\|_{L^\infty}^2 + \|u_{0,x}\|_{L^\infty}^2)e^{[\frac{1}{2}+\frac{1}{2}k_3^2(T)]t}$.

Thus,

$$\left|\sup_{x\in\mathbb{S}} u_x(t,x)\right| = \left|M(t)\right| \leqslant \frac{\nu(t)}{\gamma(0)} \leqslant \frac{(1 + \|\rho_0\|_{L^\infty}^2 + \|u_{0,x}\|_{L^\infty}^2)}{\inf_{x\in\mathbb{S}} \rho_0(x)} e^{[\frac{1}{2}+\frac{1}{2}k_3^2(T)]t}. \tag{5.17}$$

Assume on the contrary that $T < \infty$ and the solution blows up in finite time. It then follows from Theorem 3.1 that

$$\int_0^T \left\|u_x(t,x)\right\|_{L^\infty} dt = \infty. \tag{5.18}$$

By (5.7), (5.11), (5.15), and (5.17), we have

$$\left|u_x(t,x)\right| < \infty, \quad \forall (t,x) \in [0,T) \times \mathbb{S},$$

which is a contradiction of (5.18). Thus, $T = +\infty$ and the solution $X = \binom{u}{\rho}$ is global. This completes the proof of Theorem 5.1. $\square$

If $\sigma = 0$, then we can rewrite (1.1) as follows:

$$\begin{cases} u_{txx} - \rho\rho_x + Au_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), \quad \rho(t, x+1) = \rho(t, x), & t \geqslant 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.19}$$

Next we will show that the solutions to system (5.19) are global-in-time.

Theorem 5.2. *Let $\sigma = 0$. Given any $X_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geqslant 2$, there exist a maximal $T = \infty$, and a unique solution $X = \binom{u}{\rho}$ to (5.19) such that*

$$X = X(\cdot, X_0) \in C\big([0, \infty); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})\big) \cap C^1\big([0, \infty); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})\big).$$

Moreover, the solution depends continuously on the initial data.

Proof. To prove this theorem of global well-posedness of solutions to (5.19), we need the estimates for u_x in Lemma 3.1 and Theorem 3.1. Assume on the contrary that $T < \infty$ and the solution blows up in finite time. It then follows from Theorem 3.1 that

$$\int_0^T \big\| u_x(t, x) \big\|_{L^\infty} \, dt = \infty. \tag{5.20}$$

However, from Lemma 3.1, we have

$$\big| u_x(t, x) \big| < \infty, \quad \forall (t, x) \in [0, T) \times \mathbb{S},$$

which is a contradiction of (5.20). Thus, $T = +\infty$, and the solution $X = \binom{u}{\rho}$ is global. This completes the proof of Theorem 5.2. $\square$

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